Abstract

We study the term structure of interest rates in the presence of consumption commitments using an equilibrium model. Under reasonable assumptions we prove the existence and uniqueness of the equilibrium and develop computation methods. Examples are analyzed to illustrate the effect of consumption commitments on the term structure and its manifestations.
1 Introduction

The term structure of interest rates has long been one of the most active areas of research in financial economics. This is due to the importance of the term structure in pricing interest derivatives, in managing bond portfolios and in anticipating the status of the economy.

We are investigating a general equilibrium model for determining the term structure of interest rates. One of the emphases is to explicitly model and analyze the impact of consumer consumption commitments on the term structure of interest rates. Common instances of such consumption commitments are, for example, the anticipated expenses of a dependent child’s education or a consumer’s retirement savings. Other examples include an insurance company’s projected need for funds due to their insurance portfolio or a foreign country’s need for capital in specific time periods. In this investigation, we generalize the discrete infinite period Lucas general equilibrium model [Lucas, 1978] to investigate this problem. Similar to the Lucas model, we assume one representative producer and one representative consumer. To describe the distribution of the product, we represent the producer as one share of stock and the production of the producer is represented by a stationary Markov dividend process which is the sole source of uncertainty. Different from the Lucas model, we need to introduce a government that issues bonds of various durations. Moreover, the consumption commitments are described as an array whose components indicate the consumption commitments in the corresponding period. Finally, we assume that the consumption commitments are ‘soft’ so that they can be left unsatisfied. We penalize the unsatisfied consumption commitments by a concave penalty function.

For this model, we derive existence and uniqueness theorems for the equilibrium solutions under verifiable conditions and we also develop numerical procedures for computing the equilibrium. An equilibrium solution here is a yield function of three variables $y = y(D, i, t)$ where $D$ represents the states of the economy in terms of the dividend of the ‘stock’, $i$ indicates the term of the bond and $t$ signifies the period. Using a combination of numerical and analytical methods we explore the qualitative properties of the resulting yield functions. We find that qualitatively, this model can explain many of the phenomena observed in the bond yield such as those summarized in the expectation theory and the market segmentation theory. Furthermore, the impact of consumption commitments indicated by this model is compatible with intuition.
By and large, existing term structure models can be categorized into short rate models and no arbitrage models. Vasicek [Vasicek, 1977] pioneered the short rate models. The essence of this theory is to assume that short rate evolution is known. Using the short rate as input, one determines the yield curve according to the principle of consistency – no arbitrage is allowed. Early research in this direction focused on the simplicity of the models and their explicit yield curve formulae. Although the theory has been developed to much sophistication, this tradition of pursuing simplicity continues (see e.g. [Cox, Ingersoll, and Ross, 1985, Chan et al., 1992, Diffie and Kan, 1996]). Similarly, the no arbitrage models are based on the principle of no arbitrage as the name suggests. The difference is that they take the current yield curve as a starting point. The Ho and Lee model [Ho and Lee, 1986] and Hull and White model [Hull and White, 1990] are representatives of early works in this direction. Heath, Jarrow and Merton’s paper [Heath, Jarrow, and Merton, 1992] provided a general framework for this methodology. Both methods have been developed considerably since. More recently, Cairns’ book [Cairns, 2004] gave an accessible introduction and explored many newer developments in this vast field. The equilibrium approach that we use here is more general. Once we found an equilibrium \( y = y(D, i, t) \), fixing \( t = 0 \), we have the corresponding current yield curve to all the possible states. Fixing \( i = 1 \), the function gives us the future short rate. In gaining such generality, the tradeoff is, at least in this initial investigation, the loss of some sophistication in the models. Thus, the analysis and numerical exploration here is more qualitatively oriented. Moreover, simplicity in terms of explicit or quasi-explicit formulae is no longer expected. This is, in our opinion, a worthy tradeoff. The computational power has developed significantly in the several decades since the time of those pioneering studies on term structure. Delegating some of the task to computers is not only feasible but also preferred. However, doing so often means that we need to delve deeper into the qualitative behaviors of the model first.

Treating consumption commitments as soft constraints through a penalty function adds a bit of technical complication to the original Lucas model. Instead of just one utility function, we now have to deal with a utility function, a penalty function and their respective interactions. Involving bonds into the model further changes its nature. The stock in the model, representing the producer, lives indefinitely. Bonds, on the other hand, live only a finite period determined by its term. Moreover, as we move from one period to the next, the bonds have the term shifting property that the terms of all
the bonds reduce by one, with the one period bonds exiting. Finally and most importantly, when in equilibrium, all the stock and bonds will be held. Feasibility constraints will cause the one share of stock held in all the periods to cancel out, leaving only the dividend associated with the stock in the equilibrium quantity. As the dividend is exogenous and Markovian, in the absence of bonds, every period would look the same, and using dynamic programming one needs only to focus on one typical period. This is no longer the case once bonds are involved. In general, every period looks different. To avoid the technical difficulties in dealing with infinitely many different periods in the dynamic programming equations, we focus on the case in which the consumption commitments and the distribution of outstanding bonds are different only for a finite number of periods. These are reasonable restrictions. In practice, one is mostly concerned with the planning for the finite future. With these restrictions we can use dynamic programming methods to derive a finite number of equations - a somewhat more complicated process than the one period problem in the Lucas model but still tractable. The involvement of bonds also leads to another complication. Due to the term shifting property alluded to previously, the value of the bonds will generally become an indispensable part of the equilibrium quantity. Thus, the model involves the bond prices in a more complicated way as compared to the stock prices. As a result, the issues of the existence of equilibrium solutions and their computation become much more involved. Fortunately, the existence of equilibriums can be established under reasonably clean conditions by using Schauder’s fix point theorem. However, ensuring the uniqueness of the equilibrium and computing it requires more delicate treatment. In our model, ensuring the equilibrium to be a fixed point of a contraction still seems to be the most reasonable approach. To achieve this, we need a delicate coupling of conditions involving the utility function, the penalty function and the discount factor.

The Lucas model has been extended to include bonds by Judd, Kubler and Schmedders in [Judd, Kubler, and Schmedders, 2003, Judd, Kubler, and Schmedders, 2006, Kubler and Schmedders, 2003]. The emphases in those studies were issues related to Perato inefficiency of the competitive equilibrium, investor’s portfolio separation and trading volume of the bonds. The model in [Judd, Kubler, and Schmedders, 2006, Judd, Kubler, and Schmedders, 2006, Kubler and Schmedders, 2003] assumed that the agents could both purchase and issue bonds and that the quantity of bonds netted to 0. This necessitated the consideration of multiple agents, which generalized the Lucas model in a different direction. By treating gov-
ernment (the bond issuer) exogenously to the model, we are able to use only one representative agent as in the original Lucas model [Lucas, 1978]. This brings us computational tractability so that we can delve deeper into the qualitative properties of the model with the assistance of numerical methods.

The rest of the paper is arranged as follows. The model is explained in detail in Section 2. We then discuss dynamic programming equations for the model and its general properties. Section 4 focuses on the tractable case in which there are only a finite many non-uniform consumption commitments. In this special case, we derive the existence of the equilibriums. We then follow up on uniqueness conditions for the equilibrium and their related computational methods in Section 5. In Section 6, we discuss examples and qualitative behaviors of the yield functions related to various input parameters. We conclude in Section 7. Longer technical proofs are contained in the Appendix.

2 The model

We consider an exchange economy with infinite period $t = 0, 1, 2, \ldots$ in which producers produce to meeting the need of consumers and the government. Besides taxation, the government issues bonds of various maturities to finance its operation. We take the production and government financing decisions as exogenous to our model. We are interested in the term structure of the interest rates in an exchange equilibrium and our main focus will be the impact of pre-committed consumption of consumers on the term structure. We generalize the Lucas general equilibrium model [Lucas, 1978] to investigate this problem. As in the Lucas model we assume a single representative consumer. The consumer make consumption decision in each period and adjust his/her investments in bonds and equity. We use a sequence $\tilde{\gamma} = (\gamma_0, \gamma_1, \ldots)$ to characterize consumer's consumption commitments at different time periods. Consumption plans that fell short in meeting those commitments will be penalized. In the spirit of the Lucas model we assume a representative producer who produce a single perishable good which is set as numeral. The government operation is mainly supported by taxation proportional to the total production and supplemented by issuing and maintaining a complete set of bonds with durations ranging from 1 to $m$ periods proportional in par value to the total production. Thus, we can conveniently think the producer
as one unit share of stock which has a markovian dividend process for net consumer consumption. A more precise description of the model follows.

The producer can be regarded as one unit share of stock whose products in each period can be regarded as a dividend process \( D_t \). We assume that \( D_t \) is a stationary markovian process defined by its transition function

\[
F(D', D) = \text{Prob}(D_{t+1} \leq D' \mid D_t = D).
\] (2.1)

The ex-dividend price at \( t \) of the stock is denoted by \( s_t \).

\( D_t \) is the sole source of uncertainty in this model. We make the following technical assumption on \( D = D_t \).

(A1) The dividend \( D_t \) has a compact positive range \([D_m, D_M]\) and that the mapping \( T : C([D_m, D_M], \mathbb{R}^m) \to C([D_m, D_M], \mathbb{R}^m) \) defined by

\[
(Tg)(D) = \int g(D')dF(D', D)
\]

maps any bounded set in \( C([D_m, D_M], \mathbb{R}^m) \) to an equi-continuous subset of \( C([D_m, D_M], \mathbb{R}^m) \).

Here \( C([D_m, D_M], \mathbb{R}^m) \) signifies the Banach space of all continuous vector functions from \([D_m, D_M]\) to \( \mathbb{R}^m \) with the sup norm \( \|f\| = \sup_{t \in [D_m, D_M]} \|f(t)\|_\infty \), where \( \| \cdot \|_\infty \) signifies the maximum norm for vectors in \( v \in \mathbb{R}^m \) defined by \( \|v\|_\infty = \max\{|v_i| : i = 1, 2, \ldots, m\} \). Condition (A1) is automatically satisfied when \( D \) has only a finite number of positive states.

The government fund its operation mainly through taxation at a fixed portion \( qD_t \) of the dividend at \( t \), where \( q \in (0, 1) \). The remainder of the dividend \((1 - q)D_t\) is the consumer’s portion which has the same transition function as that of \( D_t \). The government also maintains a complete array of (zero coupon) bond with duration 1 through \( m \) periods and each will have a fixed price 1 at maturity. The bonds are used to finance additional government consumption beyond taxation. At \( t \), the one-period bond issued at \((t-1)\) expires. The \( k > 1 \) period bond at \((t-1)\) becomes a \((k-1)\)-period bond and the government will issue a new \( m \) period bond to maintain a complete set of the bonds. The price of the \( i \)-period bond at time \( t \) is denoted \( p_t(i, D_t) \).

Since we are dealing with real rates the bond price may exceed 1. To simplify notation we use price (column) vector \( p_t(D_t) = (p_t(1, D_t), \ldots, p_t(m, D_t))^\top \) to represent the prices of the \( m \) bonds in the economy at \( t \) and we will omit the variable \( D_t \) when this does not cause confusion. We will use vector
$B_t = (B_t(1), \ldots, B_t(m))^\top$ to denote the quantity of bonds at $t$. The total par value of the government bonds is assumed to be a fixed portion of the total expected production, that is

$$\bar{1} \cdot B_t = \sum_{i=1}^m B_t(i) = \alpha E[D_t] = \alpha E[D],$$

where $\alpha$ reflects the government fiscal policy towards debt. We can roughly think $\alpha$ as a proxy for the ratio of total government debt to GDP. Since $D_t = D$ is stationary, the total par value of the government bonds is independent of $t$. These bonds will be purchased by the consumer and acting as a medium for the consumer to transfer his/her consumption from one period to another. In equilibrium the relative quantity of bonds and the stock will dictate a bond price term structure.

The consumer wishes to maximize

$$E \left[ \sum_{t=0}^{\infty} \beta^t (u(c_t) + \phi(a_t - \gamma_t)) \right].$$

(2.3)

Here $c_t$ is a stochastic consumption process, $a_t$ is the stochastic process of allocation for (partially) satisfying the consumption commitment, $\beta$ is a discount factor, $u$ and $\phi$ are the period utility function and the penalty function for deviation from the consumption commitments, respectively. The sum $c_t + a_t$ is the consumption portion from consumer’s net worth $w_t$ at time $t$. As usual we will assume that $u$ is a strictly increasing concave function reflecting the risk aversion of the agent. The penalty function $\phi(s)$ is a concave function attains its maximum at $s = 0$ and $\phi(0) = 0$ reflecting the penalty towards deviation from the consumption commitments. Insufficient allocation to meet the commitment indicated by $a_t - \gamma_t < 0$ will be penalized. Assuming $\phi$ to be concave means that as the deficit in satisfying the consumption commitment $a_t - \gamma_t$ grows larger, the rate of marginal penalty will increase. With this model there is no incentive to allocate $a_t$ above the consumption commitment $\gamma_t$ because doing so will reduce the utility and increase the penalty (reduce the value of $\phi(a_t - \gamma_t)$). Thus, additional consumption power will either be absorbed by current consumption $c_t$ or put into saving for future consumption. On the other hand, usually $a_t - \gamma_t < 0$. When $\gamma_t = 0$ this means that we will allow $a_t$ to take negative value. The consumer distributes his consumption power among different time periods by holding
and adjusting a portfolio of the stock and bonds. We assume that the adjustment of the portfolio can only be made at the beginning of a period and use \((\theta_t, \psi_t) = (\theta_t, \psi_t(1), \ldots, \psi_t(m))\) to represent the consumers portfolio at time \(t\), in which \(\theta_t\) represents the share of stocks and \(\psi_t(i), i = 1, 2, \ldots, m\) represents the share of the \(i\) period bond. We note that, in general, \(c_t, a_t, \theta_t\) and \(\psi_t\) are all depend on \(D_t\).

**Constraints:** We do not allow negative consumption so that

\[
c_t \geq 0.
\] (2.4)

Clearly, we need to impose a budget constraint that total value of consumption and savings in the form of a portfolio for the next period cannot exceed the current net worth of the agent \(w_t\), i.e., for \(t = 0, 1, \ldots,\)

\[
c_t(D_t) + a_t(D_t) + \theta_t(D_t)s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) \leq w_t,
\] (2.5)

where, \(w_0\) is given and, for \(t = 1, 2, \ldots,\)

\[
w_t = \theta_{t-1}(D_{t-1})(s_t(D_t) + (1 - q)D_t) + \psi^1_{t-1}(D_{t-1}) + \psi^2_{t-1}(D_{t-1}) + \cdots + \psi^m_{t-1}(D_{t-1})p^m_{t-1}(D_t).
\]

Alternatively, we can also write \(w_t\) in vector form as

\[
w_t = \theta_{t-1}(D_{t-1})(s_t(D_t) + (1 - q)D_t) + \psi_{t-1}(D_{t-1}) \cdot (A p_t(D_t) + b),
\]

where \(A\) and \(b\) are defined as follows

\[
A = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

The alternative representation reveals that \(w_t\) is an affine function of the stock and bond prices. The consumer is not allow to short or issue bonds so that \(\psi_t(i) \in [0, B_t(i)]\) for all \(t\) and \(i\).

**Equilibrium quantities:** Since the consumption commitment \(\gamma\) is a soft constraint from which one can deviate, we always have the following feasible quantitative equilibrium: The consumer holds all the shares of stock and bonds at all time and consumes all the after-tax dividend of the stock and interest of the bonds in each period. That corresponds to, for all \(t,\)

\[
\theta_t = 1, \psi_t = B_t
\]
\[ c_t(D_t) + a_t(D_t) + B_t \cdot p_t(D_t) - B_{t-1} \cdot (A p_t(D_t) + b) + q D_t = D_t, \]

or
\[ c_t(D_t) + a_t(D_t) = (1 - q) D_t + B_{t-1} \cdot (A p_t(D_t) + b) - B_t \cdot p_t(D_t). \]

Here \( g_t = B_t \cdot p_t(D_t) - B_{t-1} \cdot (A p_t(D_t) + b) \) is the government’s additional expenditure financed by issuing bonds. When \( g_t > 0 \), the government consumes additional \( g_t \) from the consumer’s portion of the dividend and when \( g_t < 0 \) the government yields part of its portion of the dividend for consumers. Our interest is what price process \( p_t \) (which determines the term structure of interest rates) will induce such a general market equilibrium and what are the impact of the consumption commitment \( \vec{\gamma} \) on this term structure.

We assume that the agent is rational and, therefore, his/her portfolio strategy is driven by maximizing the expected utility (2.3) subject to the constraints in (2.4) and (2.5), i.e.,

\[
\begin{align*}
\text{maximize} & \quad E \left[ \sum_{t=0}^\infty \beta^t [u(c_t) + \phi (a_t - \gamma_t)] \right] \\
\text{subject to} & \quad c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, t = 0, 1, \ldots, \\
& \quad c_t(D_t) + a_t(D_t) + \theta_t(D_t) s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) \\
& \quad \quad \leq \theta_{t-1}(D_{t-1}) (s_{t-1}(D_t) + (1 - q) D_t) \\
& \quad \quad + \psi_{t-1}(D_{t-1}) \cdot (A p_t(D_t) + b), t > 0, \\
& \quad c_0(D_0) + a_0(D_0) + \theta_0(D_0) s_0(D_0) + \psi_0(D_0) \cdot p_0(D_0) \leq w_0.
\end{align*}
\]

The assumption \( \psi_t(i) \in [0, B_t(i)] \) means that the consumer is not allowed to issue bonds. We will always assume this constraint below. On the other hand, we don’t make such restrictions on the stock so that the consumer can short stock or trade stock on margin. We assume that the initial state \( D_0 \) and the initial endowment \( w_0 \) are fixed.

### 3 Dynamic programming

#### 3.1 Dynamic programming equations

The optimal value of the problem (2.6) depends on the dividend \( D = D_0 \), initial wealth \( w = w_0 \), the consumption commitments \( \vec{\gamma} = (\gamma_0, \gamma_1, \ldots) \). We
denote it by $v(w, D, \gamma)$. Using $S$ to signify the shifting operator on an infinite sequence $\bar{\alpha} = (a_0, a_1, \ldots)$ defined by $S\bar{\alpha} = (a_1, a_2, \ldots)$ then the well known optimality principle tells us that $v$ is the optimal value function of problem (2.6) if and only if it satisfies the following system of dynamic programming equations, for $t = 0, 1, \ldots,$

$$v(w_t, D_t, S^t \gamma) = \max \left[ u(c_t) + \phi(a_t - \gamma_t) + \beta E[v(w_{t+1}, D_{t+1}, S^{t+1} \gamma)|D_t = D] \right]$$

subject to

$$c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m,$$

$$c_t + a_t + \theta_t s_t(D_t) + \psi_t \cdot p_t(D_t) \leq w$$

$$\theta_t(s_{t+1}(D_{t+1}) + (1 - q)D_{t+1}) + \psi_t \cdot (A p_{t+1}(D_{t+1}) + b) = w_{t+1}.$$ 

The dividend process $D = D_t$ is markovian and the wealth level $w = w_t$ can change. Thus, generically we can use the distribution (2.1) to rewrite the dynamic programming equations as, for $t = 0, 1, \ldots,$

$$v(w, D, S^t \gamma) = \max \left[ u(c_t) + \phi(a_t - \gamma_t) + \beta \int v(w', D', S^{t+1} \gamma)dF(D', D) \right]$$

subject to

$$c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m,$$

$$c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w$$

$$\theta_t(s_{t+1}(D') + (1 - q)D') + \psi_t \cdot (A p_{t+1}(D') + b) = w'.$$

Next we assume that the price processes $\bar{s} = (s_0, s_1, \ldots)$ and $\bar{p} = (p_0, p_1, \ldots)$ are known and view problems (3.2) as portfolio maximization problems. It is then helpful to write the value function as a function of the portfolio. Due to the relationship

$$\theta_t(s_t(D_{t+1}) + (1 - q)D_{t+1}) + \psi_t \cdot (A p_{t+1}(D_{t+1}) + b) = w_{t+1}$$

we naturally define

$$V(\theta, \psi, D, \gamma|\bar{s}, \bar{p}) = v(\theta(s_0(D) + (1 - q)D) + \psi \cdot (A p_0(D) + b), D, \gamma).$$

We emphasis that

1. the portfolio $(\theta, \psi)$ is independent on $D_{t+1},$ since it is generated by the optimization problem for the period $t$ and depends only on $D = D_t,$ and
2. the price processes $\tilde{s}$ and $\tilde{p}$ relates the portfolio and the wealth level and, therefore, they must be specified in the definition of $V$.

Using the value function $V$ we can write the dynamic programming equations (3.2) as, for $t = 0, 1, \ldots$,

$$V(\theta, \psi, D, S^t | S^t \tilde{s}, S^t \tilde{p}) =$$

$$\max \left[ u(c) + \phi(a - \gamma_0) + \beta \int V(\theta', \psi', D', S^{t+1} | S^{t+1} \tilde{s}, S^{t+1} \tilde{p}) dF(D', D) \right]$$

subject to

$$c \geq 0, \psi'(i) \in [0, B_t(i)], i = 1, 2, \ldots, m,$$

$$c + a + \theta_s(D) + \psi \cdot p_t(D) \leq \theta(s_t(D) + (1 - q)D) + \psi \cdot (A p_t(D) + b).$$

**Remark 3.1.** Strictly speaking, $v$ and, therefore, $V$ also depend on $B_t$. To simplify the notation we will omit $B_t$ in their specification.

### 3.2 Differentiability of the value function

The following properties of the value function $v$ generalize those of [Lucas, 1978, Prop. 2] and will be helpful below. The proof is similar, we include it in the Appendix for completeness.

**Proposition 3.2.** Fix $D$ and $\tilde{\gamma}$. Suppose that the maximum in (3.2) is attained at a positive $c = c(w)$. Then $w \rightarrow v(w, D, \tilde{\gamma})$ is differentiable and

$$v_w(w, D, \tilde{\gamma}) = u'(c) = \phi'(a - \gamma_0).$$

Moreover $w \rightarrow v(w, D, \tilde{\gamma})$ is an increasing concave function.

The condition $c(w) > 0$ in Proposition 3.2 is not too restrictive. For example, if we use the log utility function $u(c) = \ln c$, then it is automatically satisfied.

The conclusion $u'(c) = \phi'(a - \gamma_0)$ comes up due to the involvement of penalty function. Intuitively, this relationship tells us that the split of the total consumption $c + a = x$ should occur at where the margin of increase in the utility function $u$ and the penalty function $\phi$ balances. This indicates a certain ‘coupling’ of the properties of $u$ and $\phi$. An easy (yet useful) sufficient condition that ensures such a split is recorded as a lemma below.
Lemma 3.3. Suppose that, for each $x > \gamma > 0$, there exists a $\xi > x - \gamma$ such that

$$u'(\xi) < \phi'(x - \gamma - \xi).$$

Then, the equations

$$u'(c) = \phi'(a - \gamma)$$
$$c + a = x.$$  \hfill (3.5)

uniquely determine $c = c(x - \gamma) \in [x - \gamma, \xi]$. Moreover,

$$c'(x - \gamma) = \frac{\phi''(x - \gamma - c(x - \gamma))}{\phi''(c(x - \gamma)) + \phi''(x - \gamma - c(x - \gamma))} > 0$$

and, therefore, the function $c$ is increasing.

Proof. The system of equations in (3.5) is equivalent to

$$0 = f(c, x) = u'(c) - \phi'(x - \gamma - c).$$

It is easy to see that the continuous function $c \to f(c, x)$ is strictly decreasing, $f(x - \gamma, x) = u'(x - \gamma) > 0$ and, by the assumption of the lemma, $f(\xi, x) < 0$. Thus, the system of equations (3.5) uniquely determines $c = c(x - \gamma) \in [x - \gamma, \xi]$. The computation formula for $c'(x - \gamma)$ follows from the implicit function theorem. Q.E.D.

4 A special case: Finite number of different consumption commitments

Here we concentrate on the finite number of different consumption commitments which has better tractability. In practice, market agents are likely only concerned with a finite time horizon and, therefore, naturally focus on a finite number of different consumption commitments.

Suppose all the different consumption commitments are within $n$ periods so that $S^t\gamma = \gamma \mathbf{1}$ for $t = n, n + 1, n + 2, \ldots$. We also assume that $B_t = B_{n-1}$, for all $t \geq n - 1$. As a result the dynamic programing equations contain only $n + 1$ different equations. They are, for $t = 0, 1, \ldots, n - 1$, 

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\[ v(w, D, S^{t+1}) = \max \left[ u(c_t) + \phi(a_t - \gamma_t) + \beta \int v(w', D', S^{t+1}) dF(D', D) \right] \]

subject to
\[
\begin{align*}
    c_t \geq 0, & \quad \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, \\
    c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w \\
    \theta_t (s_{t+1}(D') + (1 - q)D') + \psi_t \cdot (A_{t+1}(D') + b) = w'.
\end{align*}
\]

and
\[ v(w, D, \gamma_1) = \max \left[ u(c_n) + \phi(a_n - \gamma_n) + \beta \int v(w', D', \gamma_1) dF(D', D) \right] \]

subject to
\[
\begin{align*}
    c_n \geq 0, & \quad \psi_n(i) \in [0, B_n(i)], i = 1, 2, \ldots, m, \\
    c_n + a_n + \theta_n s_n(D) + \psi_n \cdot p_n(D) \leq w \\
    \theta_n (s_{n+1}(D') + (1 - q)D') + \psi_n \cdot (A_{n+1}(D') + b) = w'.
\end{align*}
\]

The last equation is different since \( S^{t+1} = \gamma_1, t \geq n \). Since the dividend process is markovian, we can expect \( S^n p = S^{n+1} p \) and \( S^n s = S^{n+1} s \).

### 4.1 Uniqueness of the value function

Dynamic programming equations (4.1) and (4.2) uniquely determine a bounded value function for problem (2.6) when \( \gamma_1 = \gamma \) for all \( n, n+1, \ldots \).

**Proposition 4.1.** Let \( \gamma \) satisfy \( \gamma_t = \gamma, t = n, n+1, \ldots \). Then equations (4.1) and (4.2) uniquely determine a set of bounded continuous functions \( v(w, D, S^t), t = 0, 1, \ldots, n \), such that \( v(w, D, \gamma_t) \) is the optimal value function of problem (2.6).

**Proof.** Functions \( v(w, D, S^t), t = 0, 1, \ldots, n - 1 \) can be uniquely defined consecutively if a bounded continuous function \( v(w, D, \gamma_1) \) is given. Thus, we need only to show that equation (4.2) determines a unique bounded continuous function \( v(w, D, \gamma_1) \). This follows from the Banach fixed point theorem in a manner similar to the proof of [Lucas, 1978, Prop 1]. Q.E.D.

For given continuous price processes \( \bar{s} \) and \( \bar{p} \), we can also write the dynamic programming equations (4.1) and (4.2) in terms of the value function \( V \) as, for \( t = 0, 1, \ldots, n - 1 \),
\[
V(\theta, \psi, D, S^t \gamma | S^t \bar{s}, S^t \bar{p}) = \\
\max [u(c_t) + \phi(a_t - \gamma_t) + \beta \int V(\theta', \psi', D', S^{t+1} \gamma | S^{t+1} \bar{s}, S^{t+1} \bar{p}) dF(D', D)]
\]
subject to
\[
c_t \geq 0, \psi'(i) \in [0, B_t(i)], i = 1, 2, \ldots, m,
\]
\[
c_t + a_t + \theta s_t(D) + \psi' \cdot p_t(D) \leq \theta(s_t(D) + (1 - q) D) + \psi \cdot (A p_t(D) + b).
\]

and
\[
V(\theta, \psi, D, \gamma^T | S^n \bar{s}, S^n \bar{p}) = \\
\max [u(c_n) + \phi(a_n - \gamma_n) + \beta \int V(\theta', \psi', D', \gamma^T) dF(D', D)]
\]
subject to
\[
c_n \geq 0, \psi'(i) \in [0, B_n(i)], i = 1, 2, \ldots, m,
\]
\[
c_n + a_n + \theta s_n(D) + \psi' \cdot p_n(D) \leq \theta(s_n(D) + (1 - q) D) + \psi \cdot (A p_n(D) + b).
\]

As a consequence of Proposition 4.1 we have

**Corollary 4.2.** Let \( \gamma \) satisfy \( \gamma_t = \gamma \) for \( t \geq n \), and let \( \bar{s} \) and \( \bar{p} \) satisfy \( S_t p = S'_t p, S_t s = S'_t s \) for \( t, t' \geq n \). Then equations (4.3) and (4.4) uniquely determine a set of bounded continuous functions

\[
V(\theta, \psi, D, \gamma^T | S^t \bar{s}, S^t \bar{p}), t = 0, 1, \ldots, n,
\]
such that \( V(\theta, \psi, D, \gamma^T | \bar{s}, \bar{p}) \) is the optimal value function of problem (2.6) as an optimal portfolio problem for given price processes \( \bar{s} \) and \( \bar{p} \).

### 4.2 Equilibrium

An equilibrium price is one in which all the bonds and the stock are held by the consumer.

**Definition 4.3.** Let \( \gamma \) be a consumption commitment with \( \gamma_t = 0 \) for \( t \geq n \). An equilibrium is a pair of stock and bond prices processes that \( \bar{s}, \bar{p} \) and a continuous bounded optimal value function \( V(\theta, \psi, D, \gamma^T | \bar{s}, \bar{p}) \) satisfying...
(i) $S^t \vec{p} = S^{t'} \vec{p}, S^t \vec{s} = S^{t'} \vec{s}$, for $t, t' \geq n$.

(ii) dynamic programming system of equations (4.3) and (4.4), and

(iii) for each $D$ and $t = 0, 1, \ldots, n$, $V(1, B_t, D, S^t \vec{p}, S^t \vec{s}, S^{t'} \vec{p})$ is attained by $\theta' = 1, \psi' = B_t$, and $c + a = (1 - q)D + B_{t-1} \cdot (A p_t(D) + b) - B_t \cdot p_t(D)$.

Developing the methods in Lucas [Lucas, 1978], we can show the existence of an equilibrium. For this we will need one more technical assumption.

(A2) The utility function $u$ and the penalty function $\phi$ have continuous second order derivatives and there exists a constant $\delta \geq 0$ such that

$$(1 - q)D_m + \min_t B_t(1) - (1 + \delta)\alpha E[D] - \|\vec{\gamma}\|_\infty > 0,$$

and, for any $f, g \in C([D_m, D_M]; R^m)$ with $\|f\|, \|g\| \leq 1 + \delta$,

$$\beta \frac{u'(c_{t+1}(D', f))}{u'(c_t(D, g))} \leq 1, \forall D, D' \in [D_m, D_M], t = 0, 1, \ldots n,$$

where

$$c_t(D, g) = c((1 - q)D - B_t \cdot g(D) + B_{t-1} \cdot (A g(D) + b) - \gamma_t). \quad (4.5)$$

**Theorem 4.4.** Suppose that conditions (A1) and (A2) are satisfied. We assume also that $B_t = B_{n-1}$ for all $t \geq n - 1$. Then there exists an equilibrium. Moreover, the equilibrium price processes $\vec{s}$ and $\vec{p}$ satisfy, for $t = 0, 1, \ldots, n - 1$,

$$u'(c_t(D, p_t))(s_t(D), p_t(D)) \quad (4.6)$$

$$= \beta \int u'(c_{t+1}(D', p_{t+1}))(s_{t+1}(D') + (1 - q)D', A p_{t+1}(D') + b)dF(D', D)$$

and

$$u'(c_n(D, p_n))(s_n(D), p_n(D)) \quad (4.7)$$

$$= \beta \int u'(c_n(D', p_n))(s_n(D') + (1 - q)D', A p_n(D') + b)dF(D', D).$$

Finally, for all $D \in [D_m, D_M]$,

$$p_t(i, D) \in [0, 1 + \delta], i = 1, 2, \ldots, m, t = 0, 1, \ldots, n. \quad (4.8)$$
The idea of the proof of Theorem 4.4 comes from [Lucas, 1978]. Equations (4.7) and (4.6) are necessary and sufficient conditions for the portfolio optimization problems (4.3) and (4.4) corresponding to the portfolios \( \theta_t = 1, \psi_t = B_t \) for \( t = 0, 1, \ldots, n \). If we can show they have a solution \( (s_t(D), p_t(D)), t = 0, 1, \ldots, n \) then Corollary 4.2 ensures \( (s_t(D), p_t(D)), t = 0, 1, \ldots, n \) to be the equilibrium prices. Now we approach the key step of showing a solution \( (s_t(D), p_t(D)), t = 0, 1, \ldots, n \) exists. This is where we differ from Lucas [Lucas, 1978] which needs only to deal with a one step equation that is simpler than (4.7). In [Lucas, 1978], for the existence of an equilibrium solution, a Banach Fixed Point argument suffices. Using such an argument based on the Banach Fixed Point Theorem here would require rather restrictive conditions on the utility function \( u \) and the penalty function \( \phi \). Fortunately, it turns out that under condition (A1) and (A2) we can use Schauder’s fixed point theorem to show the existence of solutions \( (s_t(D), p_t(D)), t = 0, 1, \ldots, n \) recursively. The details of this technical proof is contained in the Appendix.

5 Computing equilibrium prices for the stock and bonds

Although the existence of equilibrium prices for the stock and bonds were established in Theorem 4.4, these results do not directly offer computation procedures for those equilibrium prices. This is because the Schauder’s fixed point theorem used in establishing the existence of solutions to equation (4.7) is a pure existence result that do not accompany a computation procedure. We further discuss how to compute the equilibrium prices for the stock and bonds in this subsection. First rewriting equations (4.6) and (4.7) by separating the stock price and bond price equations, we have

\[
 u'(c_t(D, p_t))s_t(D) = \beta \int u'(c_{t+1}(D', p_{t+1}))(s_{t+1}(D') + (1-q)D')dF(D', D) \quad (5.1)
\]

\[
 u'(c_t(D, p_t))p_t(D) = \beta \int u'(c_{t+1}(D', p_{t+1}))(Ap_{t+1}(D') + b)dF(D', D) \quad (5.2)
\]

\[
 u'(c_n(D, p_n))s_n(D)\beta \int u'(c_{n}(D', p_{n}))(s_{n}(D') + (1-q)D')dF(D', D) \quad (5.3)
\]
and

\[ u'(c_n(D, p_n))p_n(D) = \beta \int u'(c_n(D', p_n))(Ap_n(D') + b)dF(D', D). \tag{5.4} \]

Knowing \( p_n \), condition (A2) implies that the right hand side of (5.3) is a contraction for \( u'(c_n(D, p_n))s_n(D) \) under maximum norm. Thus, \( u'(c_n(D, p_n))s_n(D) \) can be computed using the iterative algorithm for the fixed point of a contraction. Since \( u'(c_n(D, p_n)) > 0 \) this uniquely determines \( s_n(D) \). The same can be said about (5.1). Thus, we focus on (5.4) and (5.2). Since the fixed point of a contraction mapping is unique and can easily be approximated using an iterative process we look for conditions under which \( p_n \) and \( p_t \) can be determined as such fixed points for contractions. This amounts to require that

(A3) the mappings

\[ Tg(D) = \beta \int \frac{u'(c_n(D', g(D')))}{u'(c_n(D, g(D)))}(Ag(D') + b)dF(D', D) \tag{5.5} \]

and

\[ T_tg(D) = \beta \int \frac{u'(c_{t+1}(D', p_{t+1}(D')))}{u'(c_t(D, p(D)))}(Ap_{t+1}(D') + b)dF(D', D) \tag{5.6} \]

are contractions.

When condition (A3) is satisfied the equilibrium prices of the stock and the bonds are uniquely determined and can be approximated by the iterative algorithms for finding the fixed point of a contraction. In actual computation, a continuous model must be approximated by a discrete one. Thus, we will focus on the case when \( D = D_t \) has only a finite number of states \( \{D(1), D(2), \ldots, D(k)\} \). The transition property of \( D \) from one state to another is then characterized by a \( k \times k \) transition matrix \( \Pi = (\pi_{ij}) \) with

\[ \pi_{ij} = \text{Prob}(D_{t+1} = D(j) | D_t = D(i)). \]

Now any function of \( D \) can be viewed as a \( k \)-dimensional vector. For example, we can view \( p_n(m, D) \) as a column vector

\[ p_n(m, D) = (p_n(m, D_1), p_n(m, D_2), \ldots, p_n(m, D_k))^\top. \]
For a vector $\vec{v} = (v_1, v_2, \ldots, v_k)^\top$ we will denote
\[
\text{diag}(\vec{v}) = \begin{bmatrix}
v_1 & 0 & \cdots & 0 \\
0 & v_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v_k
\end{bmatrix},
\]

Then we can reduce equations (4.7) to matrix equations:

\[
\Lambda_n s_n(D) = \beta \Pi \Lambda_n(s_n(D) + (1 - q)D) = \beta \Pi \Lambda_n s_n(D) + \beta(1 - q)\Pi \Lambda_n D,
\]

\[
p_n(1, D) = \beta \Lambda_n^{-1} \Pi \Lambda_n \vec{1}
\]

\[
p_n(2, D) = \beta \Lambda_n^{-1} \Pi \Lambda_n p_n(1, D)
\]

\[
p_n(m - 1, D) = \beta \Lambda_n^{-1} \Pi \Lambda_n p_n(m - 2, D),
\]

\[
p_n(m, D) = \beta \Lambda_n^{-1} \Pi \Lambda_n p_n(m - 1, D),
\]

where

\[\Lambda_n = \Lambda_n(D, p_n) = \text{diag}(u'(c_n(D, p_n))).\]

Knowing $p_n$, condition (A2) implies that the right hand side of (5.7) is a contraction for $\Lambda_n s_n(D)$ under maximum norm. Thus, $\Lambda_n s_n(D)$ can be computed using the iterative algorithm for the fixed point of a contraction. Since $\Lambda_n$ is invertible this uniquely determines $s_n(D)$.

After determining $(s_n, p_n)$, we set $(s_t, p_t) = (s_n, p_n)$ for all $t \geq n$. Then we can use equations (4.6) to determine $(s_t, p_t)$ for $t = 0, 1, \ldots, n - 1$. The concrete computation formula when $D$ has finite number of states are laid out below: for $t = 0, 1, \ldots, n - 1$,

\[
s_t(D) = \beta \Lambda_{t}^{-1} \Pi \Lambda_{t+1}(s_{t+1}(D) + (1 - q)D),
\]

\[
p_t(1, D) = \beta \Lambda_{t}^{-1} \Pi \Lambda_{t+1} \vec{1}
\]

\[
p_t(2, D) = \beta \Lambda_{t}^{-1} \Pi \Lambda_{t+1} p_{t+1}(1, D)
\]

\[
p_t(m - 1, D) = \beta \Lambda_{t}^{-1} \Pi \Lambda_{t+1} p_{t+1}(m - 2, D),
\]

\[
p_t(m, D) = \beta \Lambda_{t}^{-1} \Pi \Lambda_{t+1} p_{t+1}(m - 1, D).\]
Here

$$\Lambda_t = \Lambda_t(D, p_t) = \text{diag}(u'(c_t(D, p_t))).$$

Again $p_t$ can be derived iteratively using algorithm for contraction mappings provided that condition (A3) is satisfied.

The algorithms discussed above relies on condition (A3). Factors involved in (A3) include the utility function $u$, the penalty function $\phi$, the ranges of $B_t$ and $D$ and the discount factor $\beta$. It seems to us that seeking general sufficient conditions is less practical. Let us examining a concrete example that we will further illustrate in the next section.

**Example 5.1.** A convenient factor is that when the utility function $u$ and the penalty function $\phi$ are smooth enough, the mappings $T$ and $T_1$ defined in (5.5) and (5.6) are, in fact, Gâteaux differentiable and requiring $T$ and $T_1$ to be contractions as in (A3) is equivalent to requiring the Gâteaux derivatives $\nabla T$ and $\nabla T_1$ to have norms bounded by a constant smaller than 1.

Let us consider the concrete utility function $u(c) = \ln(c)$ and penalty function $\phi(t) = -\frac{2}{k^2}t^2$. Solving

$$u'(c) = \phi'(a - \gamma), c + a = x$$

we have

$$c(x - \gamma) = \frac{x - \gamma + \sqrt{(x - \gamma)^2 + k^2}}{2}.$$ 

Denoting

$$x_r(g, D) = (1 - q)D - B_r \cdot g(D) + B_{r-1} \cdot (Ag(D) + b) - \gamma_r$$

we have

$$u'(c_r(D, g(D))) = \frac{2}{x_r(g, D) + \sqrt{(x_r(g, D))^2 + k^2}}.$$ 

Direct computation yields

$$\nabla T = \beta \int \left( \frac{u'(c_n(D', g(D')))}{u'(c_n(D, g(D)))} (A + (Ag(D') + b)B_n^\top (A - I)} \right) \times \left( \frac{1}{\sqrt{(x_n(g, D))^2 + k^2}} - \frac{1}{\sqrt{(x_n(g, D'))^2 + k^2}} \right) dF(D', D).$$
\[ \nabla T_t = \beta \int \frac{u'(c_{t+1}(D', p_{t+1}(D'))}{u'(c_t(D, g(D)))} (Ap_{t+1}(D') + b)(B_{t-1}^T A - B_t^T) \mathrm{d}F(D', D). \]

Define
\[ \alpha = \max_{\|g\| < 1+\delta} \left\| A + (Ag(D') + b)B_n^T (A - I) \right\| \]
\[ \times \left( \frac{1}{\sqrt{(x_n(g, D))^2 + k^2}} - \frac{1}{\sqrt{(x_n(D', D))^2 + k^2}} \right) \]
and
\[ \alpha_1 = \max_{\|p_t+1\|,\|g\| < 1+\delta} \left\| (Ap_{t+1}(D') + b)(B_{t-1}^T A - B_t^T) \right\| \]
\[ \times \frac{1}{\sqrt{(x_n(g, D))^2 + k^2}}. \]

Then a sufficient condition for (A3) is, for any \( f, g \in C([D_m, D_M]; R^m) \) with \( \|f\|, \|g\| \leq 1 + \delta \),
\[ \beta \frac{u'(c_{t+1}(D', f))}{u'(c_t(D, g))} \max(\alpha, \alpha_1) < 1, \forall D, D' \in [D_m, D_M], t = 0, 1, ...n. \] \hspace{1cm} (5.16)

Since \( \max(\alpha, \alpha_1) > 1 \), condition (5.16) is a stronger requirement than (A2). This is the price we have to pay to ensure the uniqueness of the fixed points and an algorithm for calculating them.

There are several ways to reduce the quantity in the right hand side of (5.16) so as to satisfy the above sufficient condition: by reducing \( \beta \), by increasing \( k \), by reducing (scaling) \( B_t \). They are all related to modeling. Thus, condition (5.16) provides us a guide in modeling to ensure the equilibrium is unique and computable. We emphasize that (5.16) is a sufficient condition which requires more than necessary. In dealing with practical modeling problems, this sufficient condition can often be relaxed. Moreover, although condition (5.16) is derived for the specific choice of a log utility function and a quadratic penalty function, the general method can also be used to deal with other choices of utility and penalty functions.
Finally, we remark that the existence of a solution to (5.7), (5.8), (5.9) and (5.10) under condition (A2) (without any additional assumption) follows from Theorem 4.4, which uses Schauder’s fixed point theorem for mapping on an infinite dimensional space. However, when $D$ has only finite number of states as is the case in this subsection, we are actually dealing with finite dimensional vectors. In this case the existence of solution $p_t(D), t = 0, 1, \ldots, n$ can be derived directly from the Brouwer Fixed Point Theorem [Milnor, 1965] for mappings in finite dimensional spaces.

6 Examples and Discussions

The theory and computation procedures developed in the previous sections are illustrated through examples here. We also discuss economic and policy implications of these examples. To make our examples close to reality we choose our parameters to mimic the actual supply and demand situation of the US treasury bonds. According to the web site of Security Industry and Financial Market Association [http://www.sifma.net/story.asp?id=1210], the total of 4,322.86 (Billion dollar) U.S. Treasury Securities outstanding in December 2006 are distributed as follows: bills (less than 1 year) 940.77, notes (2-10 years) 2,440.47, bonds (20-30 years) 530.55 and indexed securities 411.08. Average to per year we find that the long term bonds $530.55/20 = 26$ are much less than the midterm notes $2440.47/9 = 271$ which in turn is much less than the short term bills. We use the US GDP as a proxy for the dividend in our model which is roughly 14 trillion per year. We consider a model with three states (using trillion as our unit): expanding $D(1) = 14 \cdot 1.03$, neutral $D(2) = 14$ and contracting $D(3) = 14 \cdot 0.97$. It is known that about $2/3$ of the US GDP is consumption. However, not all the consumption are elastic so we choose $1 - q = 0.3$. We use the following transition matrix

$$
\Pi = \begin{bmatrix}
0.5 & 0.3 & 0.2 \\
0.4 & 0.3 & 0.3 \\
0.5 & 0.4 & 0.1
\end{bmatrix}.
$$

We assume that the base distribution of outstanding bonds follows the actual US treasury bond data and is independent of $t$. That is, for all $t$,

$$
B_t(1) = 0.941, \quad \text{(6.1)}
$$

$$
B_t(i) = 0.271, i = 2, \ldots, 10
$$

$$
B_t(j) = 0.026, j = 11, \ldots, 30.
$$
We will use the utility function and the penalty function in Example 5.1 with $k = 10$ so that
\[ c(x - \gamma) = \frac{x - \gamma + \sqrt{(x - \gamma)^2 + 100}}{2}. \]

Next we examine the range of the parameters that are likely to satisfy the conditions (A2) and (A3). For this example
\[ x_t(D, g) = 0.3D + 0.941 - 0.67g_1 - 0.245g_{11} - 0.026g_{30} - \gamma_t. \]

Assume that $\|\gamma\|_\infty = 0.2$ and the price of the bonds do not exceed 1.01 which means $\delta = 0.01$. Then an estimate for the maximum and the minimum of $x_t(D, g)$ are $a = 5.267 = 0.3 \cdot 14 \cdot 1.03 + 0.941$ and $b = 3.864 = 0.3 \cdot 14 \cdot 0.97 + 0.941 - 1.01 \cdot (.67 + .245 + 0.026) - .2$ and, therefore, an estimate of the ratio $\frac{u'(c_{t+1}(D', f))}{u'(c_{t}(D, g))}$ is 1.136 when $\|f\|, \|g\| < 1.01$. This means to ensure (5.16) we need to choose $\beta < 0.88$, a level that is a bit unrealistic. However, as mentioned before, (A2) and (5.16) are conservative sufficient conditions. In the actual equilibrium computation the $x_t'$s involved never reach the extremes $a$ and $b$. As a result the spread between different $x_t'$s are narrower. Rather than refining the estimate of the range of the $\beta$ that works, we can simply try out a reasonable discount level. It turns out that in our current setting a realistic $\beta = 0.95$ works just fine. We will use these parameters in the tests.

### 6.1 Without consumption commitment

We first consider the special case when there is no consumption commitment or $\vec{\gamma} = \gamma \vec{1}$ with $\gamma = 0$. This special case leads to a background term structure which can be used as a yardstick to understand the impact of the consumption commitment on term structure. Using the parameters in the introduction of this section and the computation procedure discussed in the previous section we derive the yield surface in Fig. 1.

Expansion of the economy represented by the higher dividend $D(1)$ corresponding to a normal yield curve. Contraction of the economy represented by the lower dividend $D(3)$ corresponding to an inverted yield curve. In between for the neutral state $D(2)$ we see almost flat yield curves. Tests show that these configurations are rather robust. They are not sensitive to $\gamma, D, B$ and $\Pi$. It is worth noting that the term structure is sensitive to the economic status as we usually observe in the actual bond markets.

How will the consumption commitment impact the term structure?
6.2 Typical consumption commitments

A typical yield surface is drawn in Fig. 2 where we have a consumption commitment 0.15 at period 6. Since the commitment is soft, we assume it is fulfilled at period 6, 7, and 8 in equal installment corresponding to $\gamma_i = 0.05$ for $i = 6, 7, 8$. As anticipated the consumption commitments reduce the yield of the bonds of the corresponding maturity and also yield of the bonds in the vicinity.

It is interesting to know how is each consumption commitment impact the system.

6.3 One consumption commitment

Let us examine the impact of one single consumption commitment. Suppose there is a consumption commitment of 0.15 at year 8. This will certainly increase the demand for the 8 year bond and as a result reduce the amount of outstanding 8 year bond. Since action to prepare for this consumption commitment will be taken gradually and not all by buying the 8 year bond we assume that the reduction on the 8 year bond is $20\% \times 0.15$. Thus, $B_1(8) = 0.271 - 0.15 \cdot 0.2 = 0.241$ and other $B_t(i)'s$ remains the same. The
yield curves corresponding to such a single consumption commitment in Figure 3. To see the detail of the difference of these two yield surfaces we also draw their difference in Figure 4.

We observe that (i) the yield of 8 year bond is much lower than other bonds due to the consumption commitment; (ii) the yields of the other bonds are also depressed, (iii) the yield reductions for bonds with maturity close to the 8 year bonds are negatively correlated to how close they are to the 8 year bond except for (iv) the yield of the bonds with very short maturity – 1,2,3 are heavily depressed and (v) the depression of the yield for bonds with maturity shorter than 8 are heavier than those with maturity longer than 8.

Item (i) is entirely anticipated. Phenomena (ii) and (iii) are consistent with the substitution effect. Item (iv) we believe is due to the fact that in the yield computation formula

$$\frac{1}{i} \ln p_0(i)$$

the rise of the factor $1/i$ is much faster for small $i$'s. But why (v)?

Figure 2: Yield surface: consumption commitment at $t = 6$
Figure 3: One consumption commitment $\gamma_8 = 0.15$

Figure 4: Difference of Figure 3 and Figure 1
6.4 The role of liquidity

It turns out that the phenomenon described in (v) of the previous subsection can be explained by the better liquidity for the one year bond which makes substitution make more sense for those bonds that are close to the one in need. Indeed, taking away the additional liquidity for the one year bond by setting $B_t(1) = 0.271$ the same as 2-10 year bond and drawing again the difference between yield surface with a consumption commitment of 0.15 at year 8 and that of without, we get Figure 5.

Comparing Figure 5 and Figure 4 we can clearly see the role of liquidity for the 1 year bond. To confirm this observation let us put back the liquidity for the one year bond (setting $B_t(1) = 0.941$) and also add additional liquidity for the two year bond by setting $B_t(2) = 0.5$. We will again draw the difference of the yield surface derived this way and the corresponding one without consumption commitment. This gives us Figure 6. Together with Figure 5 and Figure 4, the function of the heavy supply of short term bonds is rather clear.

Similar effect can also be observed for longer term bonds. To see that let us go back to the original configuration for outstanding bonds in (6.1). This time we set a single consumption commitment of 0.015 at year 16 and again
assume the outstanding quantity of the 16 year bond is reduced by 20% of the consumption commitment \( B_1(16) = 0.026 - 0.015 \cdot 0.2 = 0.23 \). We draw the difference of yield surfaces with and without the consumption commitments in Figure 7. What is interesting here is the additional yield reduction for the 6 year bond. This is due to the abundant supply of the 2-10 year bond comparing to the longer term bonds. The consumer is buying the 6 year bond to convert into the 10 year bond when matured in anticipating good liquidity for the 10 year bond. Indeed if we make the quantity of outstanding bonds all same at say 0.026 and redraw the above difference of the yield surfaces we get Figure 8 in which all the features due to liquidity are gone.

Seeing the impact of liquidity on the yield surfaces, we wonder what will happen if the government provide enough liquidity where the extra demand is so as to keep the base structure of outstanding bonds intact?

### 6.5 Segmented market

Continue from the setting that produced Figure 8 where the bond distribution is flat at 0.26 and there is a single consumption commitment of 0.015 at year 16. Now assume the government provide additional 16 year bond so that
Figure 7: No additional liquidity for one year bond

Figure 8: Flat bond distribution
the outstanding quantity of the 16 year bond remains the same as other bonds and we get Figure 9. We see that the consumption commitment at year 16 now only influence the yield of the 16 year bond. In other words the market is completely segmented. The scenario of government provide additional liquidity to maintain the basic outstanding bond distribution in (6.1) is given in Figure 10. Again we see that beyond small substitution effect due to the additional liquidity of the one year and 10 year bonds, the impact of the consumption commitment is largely isolated due to the additional liquidity provided by the government.
7 Conclusion

By developing Lucas’ method in [Lucas, 1978], we established a general equilibrium model to analyze the impact of consumption commitments on the interest rate term structure. Under reasonable conditions we established existence and uniqueness of the equilibrium. Accompanying computation methods and algorithms are used to test this model. Those tests suggest:

1. Consumption commitments during $t$ year in the future distort the term structure by reducing the yield of the $t$ year bonds. Moreover, due to substitution effect, the yield of bonds with maturity close to $t$ years are also reduced, although to a lesser degree.

2. The term structure is the result of multiple factors. Besides consumption commitments, other major factors that influence the term structure include the status of the economy, the structure of outstanding bonds and in particular, the difference in liquidity it implies, and any additional liquidity that the government provides to meet the additional demand due to consumption commitments.
3. Higher comparative liquidity for \( t \) year bonds causes a substitution effect of period \( t \). In particular, the effect of substituting the high demand of a particular bond with adjacent bonds that we observe in real markets is due to the high liquidity of short term debt instruments.

4. Providing additional liquidity for a bond in unusually high demand by the government is an effective way of minimizing the impact of such a demand to distort the yield curve.

As usual, this investigation introduces many potential further areas of analysis. Adapting the method in this paper to develop models for actual markets would be a logical progression and an intriguing and challenging exercise. To model real markets, deeper understanding of dividend patterns and the actual consumption commitments are necessary. One can then attempt to form and test conjectures about the aggregate utility and penalty functions. Further understanding of the relationship between the equilibrium model, the short rate models and no arbitrage models is another interesting direction to pursue. The analysis of trading volumes under this model is also important in both theory and practice. We hope that the results in this paper will lay a foundation for our further investigations.

8 Appendix

The appendix contains longer technical proofs postponed from the main text.

Proof of Proposition 2.1 Assume that the maximum is attained at positive \( c = c(w) \), that is

\[
v(w, D, \vec{\gamma}) = u(c) + \phi(a - \gamma_0) + \beta \int v(w', D', S\vec{\gamma})dF(D', D),
\]

where \( w' \) is the wealth level of the next period corresponding to the optimal solution. Then, for \( h_c, h_a \) small enough we have \((c + h_c, a + h_a)\) is feasible for initial endowment \( w + h_c + h_a \). It follows that

\[
v(w + h_c + h_a, D, \vec{\gamma}) \geq u(c + h_c) + \phi(a + h_a - \gamma_0) + \beta \int v(w', D', S\vec{\gamma}')dF(D', D).
\]
Thus,

\[ v(w + h_c + h_a, D, \gamma) - v(w, D, \gamma) \geq u(c + h_c) - u(c) + \phi(a + h_a - \gamma_0) - \phi(a - \gamma_0). \]  

(8.3)

On the other hand, \((c(w + h_c + h_a) - h_c, a(w + h_c + h_a) - h_a)\) is feasible for initial endowment \(w\), which implies that

\[ v(w, D, \gamma) \geq u(c(w + h_c + h_a) - h_c) + \phi(a(w + h_c + h_a) - h_a - \gamma_0) + \beta \int v(w', D', S\tilde{\gamma})dF(D', D). \]  

(8.4)

Thus,

\[ v(w + h_c + h_a, D, \gamma) - v(w, D, \gamma) \leq u(c(w + h_c + h_a)) - u(c(w + h_c + h_a) - h_c) + \phi(a(w + h_c + h_a) - \gamma_0) - \phi(a(w + h_c + h_a) - h_a - \gamma_0). \]  

(8.5)

Setting \(h_a = 0\) in (8.3) and (8.5), dividing them by \(h_c\) and letting \(h_c \to 0\) we see that \(v\) is differentiable with respect to \(w\) and \(v_w(w, D, \gamma) = u'(c)\). Reversing the role of \(h_a\) and \(h_c\) in the above argument leads to \(v_w(w, D, \gamma) = \phi'(a - \gamma_0)\).

It is clear that \(w \to v(w, D, \gamma)\) is increasing since an increase in \(w\) expands the feasible region and, therefore, the maximum. Moreover, increasing in \(w\) provides additional financial resources. Since \(u\) is an increasing function the maximizing consumption \(c(w)\) will increase with \(w\). Thus, by (3.4), \(v_w\) is an decreasing function of \(w\), which implies that \(w \to v(w, D, \gamma)\) is concave. Q.E.D.

**Proof of Theorem 4.4.** It follows from Corollary 4.2 that, given price processes \(\tilde{s}, \tilde{p}\) satisfying \(S^t \tilde{p} = S^t \tilde{p}, S^t \tilde{s} = S^t \tilde{s}\), for \(t, t' \geq n\), there exists a unique bounded continuous function \(V\) together with \(\tilde{s}\) and \(\tilde{p}\) satisfying condition (ii) in Definition 4.3. Thus, we need only to show that the equation system (4.7) and (4.6) determine price processes \(\tilde{s}, \tilde{p}\) satisfying \(S^t \tilde{p} = S^t \tilde{p}, S^t \tilde{s} = S^t \tilde{s}\), for \(t, t' \geq n\) with optimal portfolios \(\theta_t = 1, \psi_t = B_t\) for \(t = 0, 1, \ldots, n\).

The portfolio optimization problems (4.3) are concave optimization problems. Using the Lagrange Multiplier Rule, we have the following necessary and sufficient optimality conditions

\[
\lambda(1, 1, (s_t, p_t)) = \left( u'(c_t), \phi'(a_t - \gamma_t), \beta \int V_{\theta', \psi'}(\theta', \psi', D', S^{t+1}\tilde{\gamma}|S^{t+1}\tilde{s}, S^{t+1}\tilde{p})dF(D', D) \right).
\]

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It follows that
\[ \lambda = u'(c_t) = \phi'(a_t - \gamma_t), \tag{8.6} \]
and
\[ \lambda(s_t, p_t) = \beta \int V_{\theta, \psi}(\theta', \psi', D', S'^{t+1} \bar{\gamma} | S^{t+1} \bar{s}, S^{t+1} \bar{p}) dF(D', D). \tag{8.7} \]
At the equilibrium portfolios \( \theta_t = 1, \psi_t = B_t \) for \( t = 0, 1, \ldots, n - 1 \) we have constraints
\[ c_t + a_t = x = (1 - q)D - B_t \cdot p_t(D) + B_{t-1} \cdot (Ap_t(D) + b). \tag{8.8} \]
By virtue of Lemma 3.3, conditions (8.6) and (8.8) determine
\[ c_t(D, p_t(D)) = c((1 - q)D - B_t \cdot p_t(D) + B_{t-1} \cdot (Ap_t(D) + b) - \gamma_t), \]
where \( c \) is the function defined in Lemma 3.3. By Proposition 3.2, we have
\[ V_{\theta, \psi}(\theta', \psi', D', S'^{t+1} \bar{\gamma} | S^{t+1} \bar{s}, S^{t+1} \bar{p}) = u'(c_{t+1}(D', p_{t+1}(D'))) (s_{t+1}(D') + (1 - q)D') \]
Combining with (8.7) we see that (4.6) are necessary and sufficient optimality conditions for (4.3). Similarly, we can show that (4.7) is a necessary and sufficient optimality condition for (4.4). Now we need only to show that solutions to (4.7) and (4.6) exist.
First we show that there exists a price pair \((s_n, p_n)\) satisfying (4.7). We observe that (4.7) can be written as
\[ u'(c_n(D, p_n(D))) s_n(D) \]
\[ = \beta \int u'(c_n(D', p_n(D'))) (s_n(D') + (1 - q)D') dF(D', D), \tag{8.9} \]
\[ u'(c_n(D, p_n(D))) p_n(D) \]
\[ = \beta \int u'(c_n(D', p_n(D'))) (Ap_n(D') + b) dF(D', D). \tag{8.10} \]
We then rewrite (8.10) as
\[ p_n = T(p_n) \quad (8.11) \]
where
\[ Tg(D) := \beta \int u'(c_n(D', g(D')))(Ag(D') + b)dF(D', D). \]

For \( g \in C([D_m, D_M], R^m) \) with \( \|g\| \leq 1 + \delta \) condition (A2) ensures that \( Tg \in C([D_m, D_M], R^m) \) with \( \|Tg\| \leq 1 + \delta \). Moreover, condition (A1) and the derivative formula of \( c' \) in Lemma (3.3) tell us that \( T \) maps bounded set in \( C([D_m, D_M], R^m) \) to a pre-compact (equi-continuous) subset of \( C([D_m, D_M], R^m) \).

Let \( W \) be the closed convex hull of the set
\[ T\{g \in C([D_m, D_M], R^m) : \|g\| \leq 1 + \delta\}. \]
Then \( W \) is a convex compact subset of \( C([D_m, D_M], R^m) \). Applying Schauder’s fixed point theorem [Zeidler, 1986] to mapping \( T : W \to W \) we conclude that equation (8.11) has a solution \( p_n \). Moreover, \( \|p_n\| \leq 1 + \delta \) for all \( i = 1, \ldots, m \) and \( D \in [D_m, D_M] \) so that (4.8) is verified.

Finally, given \( p_n \), the right hand side of the equation (8.9) is a contraction as an operator on \( u'(c_n(D, p_n(D)))s_n(D) \). Thus, using the Banach Fixed Point Theorem, equation (8.9) uniquely determines \( u'(c_n(D, p_n(D)))s_n(D) \) and, therefore, \( s_n(D) \).

Since for \( t \geq n \) the consumption pre-commitments are uniform, and \( B_t = B_{n-1} \) for \( t \geq n - 1 \), we set \( p_t = p_n, s_t = s_n \), for \( t \geq n \).

Knowing \((s_t, p_t)\) for \( t \geq n \), we can determine \((s_t, p_t)\) for \( t = 0, 1, \ldots, n-1 \) recursively using equations in (4.6). We need only to show the general process of deriving \((s_t, p_t)\) when \((s_{t'}, p_{t'})\), \( t' > t \) is known. The process is similar to and simpler than that of determining \((s_n, p_n)\). We rewrite equation (4.6) as

\[ u'(c_t(D, p_t))s_t(D) \quad (8.12) \]

\[ = \beta \int u'(c_{t+1}(D', p_{t+1}))(s_{t+1}(D') + (1 - q)D')dF(D', D), \]

\[ u'(c_t(D, p_t))p_t(D) \quad (8.13) \]

\[ = \beta \int u'(c_{t+1}(D', p_{t+1}))(A_{p_{t+1}}(D') + b)dF(D', D). \]
Equation (8.13) is equivalent to

\[ p_t = T_1(p_t) \tag{8.14} \]

where

\[
T_1 g(D) := \beta \int \frac{u'(c_{t+1}(D', p_{t+1}(D')))}{u'(c_t(D, g(D)))} (A p_{t+1} + b) dF(D', D).
\]

Again, applying Schauder’s fixed point theorem to mapping \( T_1 \) we conclude that equation (8.14) has a solution \( p_t \). Knowing \( p_t \), the stock price \( s_t \) can be derived using the Banach Fixed Point Theorem similar to the way that we derive \( s_n \). Q.E.D.

References


