Entropy Maximization in Finance[1]

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Abstract

We highlight the role of entropy maximization in several fundamental results in financial mathematics. They are the two fund theorem for Markowitz efficient portfolios, the existence and uniqueness of a market portfolio in the capital asset pricing model, the fundamental theorem of asset pricing, the selection of a martingale measure for pricing contingent claims in an incomplete market and the calculation of super/sub-hedging bounds and portfolios. The connection of diverse important results in finance with the method of entropy maximization indicates the significant influence of methodology of physical science in financial research.

Key words. Entropy maximization, convex duality, financial mathematics, Markowitz portfolio theory, capital market pricing model, fundamental theorem of asset pricing, hedging.

AMS classification. 52A41, 90C25, 91G99.

1 Introduction

The principle of maximum entropy appeared in statistical mechanics due to the work of Boltzmann [1] and Gibbs [11]. Statistical mechanics considers the aggregate behavior of large physical systems of microscopic elements. These aggregate behavior is the observation of the 'moments' of a probability distribution of those microscopic elements. Knowing finite number of observations there could be many different probability distributions that are consistent with these observations. The principle of maximum entropy suggests to select the probability distribution that maximizes an entropy. Jaynes' work [14, 15] relates this principle to Shannon's information theory [24]. He points out that in essence the maximum entropy methods selects the most uninformative distribution possible if one choose to use the Boltzmann-Shannon entropy.

The structures of such entropy maximization problem were explored in solving other application problem often with the Boltzmann-Shannon entropy replaced by other concave functions. This approach is referred to as entropy maximization method which has wide applications in diverse fields. We show in this paper that several important results in financial theory can be derived by using the entropy maximization method. They are the Markowitz portfolio theory and two fund theorem, the capital market pricing model, the

[1]This project started from Jonathon Borwein's comment that the two fund theorem in Markowitz portfolio theory can be viewed as exploring the structure of an entropy maximization problem and similar phenomena should exist in other financial problems. Sadly, he could not see its completion.

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fundamental theorem of asset pricing, selecting a pricing equivalent martingale measure using the entropy maximization method and determining the super/sub-hedging bounds and portfolios. The structures of the solutions to the entropy maximization problems often play a crucial role in understanding these applications.

It is not a coincidence that many financial problems can be formulated as generalized entropy maximization problems. It has been a long tradition in financial economy to model the risk aversion of a market participant using a concave utility function and assuming a rational market participant attempt to maximize his/her utility. Such a maximization problem in practice must subject to various constraints related to budget or risk control. In a simple one price economy these constraints are often linear making the resulting problem fits the pattern of an entropy maximization problem. Another modeling principle is that agents in financial market try to minimize their risk. Since diversification reduces risk, risk measures are usually convex. Thus, minimizing risk subject to various constraints also leads to generalized entropy maximization problems where the negative of the risk measure takes the role of a generalized entropy. Entropy maximization method is a special case of the more general convex duality theory (see e.g. [4, 5]). Indeed convex duality and general convex duality theory have wider applications in finance (see e.g. [6]). Nevertheless, when entropy maximization method is applicable, the structure of the entropy maximization problem and its solutions provides additional information to the financial applications.

Many important results in finance can be handled using a uniform framework of entropy maximization problems is a powerful testimony to the significant impact of physical science in financial research. This is a double edged sword. On one hand relates financial and physical models open the doors for systematically applying physical and mathematical principles and methods in financial research. This is especially beneficial in introducing effective quantitative methods into financial practice. Moreover the relationship of entropy maximization and information theory is also highly relevant in financial problems. For example, maximizing the utility of a portfolio can be interpreted as best utilize the information contained in the market model. On the other hand, we need to recognize that in some aspects financial markets are significantly different from a physical system in statistical mechanics. That means when we use theoretical results in finance, in particular, those related to the entropy maximization, caution is warranted.

The rest of the paper is arranged as follows: we lay out preliminaries regarding the entropy maximization method and a simple one period financial market model in the next section. Then we discuss four financial applications of the entropy maximization methods alluded to above in Sections 3-7. We conclude in Section 8.

2 Preliminaries

2.1 Entropy optimization problem

The mathematical formulation of an entropy maximization problem is

$$\inf_x [f(x) : Ax = b]. \tag{1}$$

Here f is a lower semicontinuous convex function representing the negative of some generalized entropy function on a Banach space X and $Ax = b$ is a linear constraint with b in a finite dimensional space representing the finite number of observations on 'moments'.

Recall that, for a lower semicontinuous convex function f on X, the Fenchel conjugate of f is defined by

$$f^*(y) := \sup_x [\langle y, x\rangle - f(x)]$$

and the subdifferential of f at x is defined by

$$\partial f(x) := \{x^* \in X^* \mid f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}.$$

Below is a concise summary of important results on duality of entropy maximization problem emphasizing the link between dual solutions and Lagrange multipliers for the primal problem (see [3, 4, 5] for details). These results are special cases of the classical convex duality theory developed by Fenchel [10], Moreau [20] and Rockafellar [22].

If constraint qualification condition (CQ)

$$b \in \text{ri } A \text{ dom } f \tag{2}$$

holds where ri signifies the relative interior then we have strong duality

$$\inf_x [f(x) : Ax = b] = \max_z [\langle z, b\rangle - f^*(A^\top z)] = (f^* \circ A^\top)^*(b). \tag{3}$$

Moreover, if $\bar{x}$ and $\bar{z}$ are solutions to the primal and dual problems, respectively, then

$$\bar{x} \in \partial f^*(A^\top \bar{z}) \tag{4}$$

$$A^\top \bar{z} \in \partial f(\bar{x}) \tag{5}$$

and

$$b \in \partial (f^* \circ A^\top)(\bar{z}). \tag{6}$$

Note that the constraint qualification condition implies the existence of a dual solution $\bar{z}$ which is the Lagrange multiplier for the primal problem. In other words, if a primal solution $\bar{x}$ exists then the Lagrangian for the primal problem

$$L(x, \bar{z}) := f(x) + \langle \bar{z}, b - Ax\rangle \tag{7}$$

as a function of x attains a minimum at $x = \bar{x}$. However, the existence of a primal solution is not always guaranteed and usually needs additional verification.

2.2 A portfolio model

To be concise we only deal with a simple one period financial market model on an economy with finite status to highlight the role of the entropy maximization methods. Many of the results discussed here also extends to more general models. We refer to books [6, 23] for details of some of the generalizations and alternative approaches.

Let $S_t = (S_t^0, S_t^1, \ldots, S_t^M), t = 0, 1$ be a financial market in a one period economy. Here $S_0^0 = 1, S_1^0 = R > 1$ represents a risk free bond and $S_t^m, m = 1, \ldots, M$ represents the price of the mth risky financial asset at time t. We assume that S_0 is a constant vector representing the prices of the assets in this financial market at $t = 0$. The risk is modeled by assuming $\hat{S}_1 = (S_1^1, \ldots, S_1^M)$ to be a random vector on a probability space $(\Omega, \mathcal{F}, P)$. A portfolio is a vector $x \in \mathbf{R}^{M+1}$ whose component x_m represents the share of the mth asset in the portfolio. Then $x \cdot S_1$ is the payoff and $x \cdot (S_1 - S_0)$ is the gain of the portfolio x both belong to $RV(\Omega, \mathcal{F}, P)$, the space of random variables on the probability space $(\Omega, \mathcal{F}, P)$. We will also use the notation $\hat{x} = (x_1, \ldots, x_M)^\top$ to denote the risky part of the portfolio.

Clearly, given a financial market S, different portfolio may corresponding to the same gain. We call such portfolios equivalent. We denote $port[S]$ the space of equivalent class of portfolios, i.e. the quotient space of $\mathbf{R}^{M+1}$ with respect to the portfolio equivalent relationship. To avoid technical complications we assume in the sequel that the sample space Ω is finite. Then it is not hard to check that the minimum norm of the portfolios in each equivalent class is a norm $\|\cdot\|_p$ for $port[S]$ and $(port[S], \|\cdot\|_p)$ is a finite dimensional Banach space. The norm $\|\cdot\|_p$ is a reasonable indication of the leverage level of a portfolio.

Having setup the model for a financial market we now turn to several important financial results that can be understood in a unified framework of entropy maximization in the next several sections.

3 Markowitz portfolio theory

Markowitz [19] considers a portfolio theory that involves only the risky assets. He postulates that the investors will want to minimize the risk given a fixed expected return and will attempt to maximize the expected return given a fixed risk. Markowitz uses the standard deviation to measure the risk of a portfolio. Standardizing the initial endowment to 1 and denote the expected return of a portfolio by μ, we can represent the Markowitz portfolio problem as an entropy maximization problem. Define

$$f(\hat{x}) = \frac{1}{2} Var(\hat{x} \cdot \hat{S}_1) = \frac{1}{2} \hat{x}^\top \Sigma \hat{x}, \tag{8}$$

where the covariant matrix

$$\begin{aligned} \Sigma &= \mathbf{E}[(\hat{S}_1 - \mathbf{E}(\hat{S}_1))^\top (\hat{S}_1 - \mathbf{E}(\hat{S}_1))] \\ &= (\mathbf{E}[(S_1^i - \mathbf{E}(S_1^i))(S_1^j - \mathbf{E}(S_1^j))])_{i,j=1,\ldots,M}, \end{aligned} \tag{9}$$

is assumed to be positive definite. Denote

$$\hat{A} = \begin{bmatrix} \mathbf{E}(\hat{S}_1) \\ \hat{S}_0 \end{bmatrix} \text{ and } b = \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{10}$$

Then we can write the Markowitz portfolio problem as

$$\min[f(\hat{x}) : \hat{A}\hat{x} = b]. \tag{11}$$

This is because minimizing f and minimizing the standard deviation σ of the payoff of the portfolio are equivalent.

Remark 3.1. Since $\sqrt{\hat{x}^\top \Sigma \hat{x}}$ can be viewed as an equivalent norm on the space of random vectors on probability space $(\Omega, \mathcal{F}, P)$ we can directly deal with all portfolios in $\mathbf{R}^M$ rather than the quotient space $port[S]$.

We can calculate that

$$f^*(y) = \frac{1}{2}\hat{y}\Sigma^{-1}\hat{y}. \tag{12}$$

It follows that letting $z = (z_1, z_2)$ we have

$$\begin{aligned} f^* \circ \hat{A}^\top(z) &= \frac{1}{2} z^\top \hat{A}\Sigma^{-1}\hat{A}^\top z \\ &= \frac{1}{2} z^\top \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} z \end{aligned} \tag{13}$$

where $\alpha = \mathbf{E}(\hat{S}_1)\Sigma^{-1}\mathbf{E}(\hat{S}_1)^\top$, $\beta = \mathbf{E}(\hat{S}_1)\Sigma^{-1}\hat{S}_0^\top$ and $\gamma = \hat{S}_0\Sigma^{-1}\hat{S}_0^\top$. It is easy to calculate that

$$\begin{aligned} (f^* \circ A^\top)^*(b) &= \max_z \left\{ z^\top \begin{bmatrix} \mu \\ 1 \end{bmatrix} - \frac{1}{2} z^\top \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} z \right\} \\ &= \frac{1}{2}[\mu, 1] \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2} = \frac{1}{2}\sigma^2. \end{aligned} \tag{14}$$

Markowitz represent each portfolio as a point in the (σ, μ)-plane. Thus, the optimal portfolio will be located on the curve

$$\sigma = \sqrt{\frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2}} \tag{15}$$

usually referred to as the Markowitz bullet due to its shape. A typical Markowitz bullet is shown in Fig. 1 with an asymptote

$$\mu = \frac{\beta}{\gamma} + \sigma\sqrt{\frac{\alpha\gamma - \beta^2}{\gamma}}. \tag{16}$$

In summary, we have

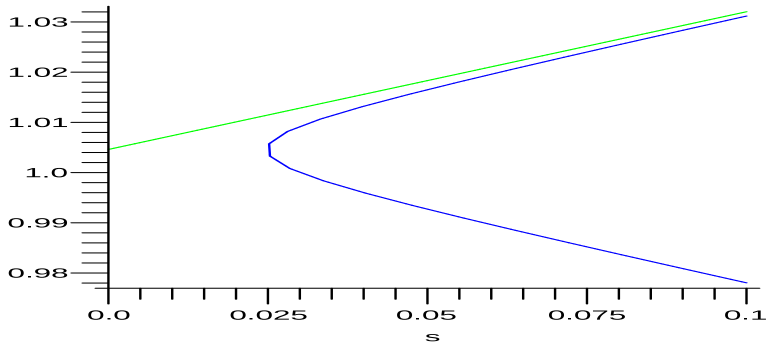

Figure 1: Markowitz Bullet

Theorem 3.2. (Markowitz Portfolio Theorem) *The effect of each portfolio $\hat{x}$ can be represented as a point in the (σ, μ)-plane. Portfolios represent optimal tradeoff between return and risk are located on the upper boundary of the Markowitz bullet given by*

$$\sigma = \sqrt{\frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2}}.$$

Remark 3.3. Markowitz portfolio problem (11) is defined on the portfolio space. The dimension of a portfolio space equals to the number of risky assets involved in the portfolio which can be quite large. For example considering the well-known benchmark SP500 index. This is a portfolio involving 500 stocks. That means considering Markowitz portfolio problem in a comparable universe of risky asset one has to deal with an entropy maximization problem in a 500 dimensional space. However, the dual problem is on a two dimensional space related to the two constraints on the expected return and the initial endowment. After standardizing the initial endowment we left with only one variable: the expected return μ. Thus, the performance of each portfolio can be intuitively represented by a point on the (σ, μ)-plane. In short, the key to the success of the Markowitz portfolio theory is to focus on the simpler dual problem (14) rather than the primal problem (11).

We now turn to discuss optimal portfolios on this Markowitz bullet. Let $\hat{\bar{x}}$ and $\bar{z}$ be the solutions to the primal and dual problems, respectively. Then it follows from (4) and (6) that

$$\hat{\bar{x}} = \Sigma^{-1}\hat{A}^\top \bar{z} \tag{17}$$

and

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \bar{z} = b = \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{18}$$

Thus,

$$\begin{aligned}\hat{x} &= \Sigma^{-1}\hat{A}^{\top}\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}^{-1}\begin{bmatrix}\mu\\ 1\end{bmatrix} \\ &= \mu\frac{\Sigma^{-1}\hat{A}^{\top}}{\alpha\gamma-\beta^2}\begin{bmatrix}\gamma\\ -\beta\end{bmatrix}+\frac{\Sigma^{-1}\hat{A}^{\top}}{\alpha\gamma-\beta^2}\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}\end{aligned} \tag{19}$$

is affine in μ. The structure of the optimal portfolio in (19) tells us that knowing two optimal portfolios one can generate any of the portfolios on the Markowitz bullet as their linear combination. This result is known as the two fund theorem.

Theorem 3.4. (Two Fund Theorem) *Select two distinct portfolios on the Markowitz efficient frontier. Then any portfolio on the Markowitz efficient frontier can be represented as the linear combination of these two portfolios.*

Proof. Let

$$\hat{x}_i = \Sigma^{-1}\hat{A}^{\top}\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}^{-1}\begin{bmatrix}\mu_i\\ 1\end{bmatrix}, i=1,2 \tag{20}$$

be two chosen portfolios on the Markowitz frontier. Suppose $\hat{x}$ is a portfolio on the Markowitz frontier. Then, for some μ,

$$\hat{x} = \Sigma^{-1}\hat{A}^{\top}\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}^{-1}\begin{bmatrix}\mu\\ 1\end{bmatrix}. \tag{21}$$

Defining

$$\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}\mu_1 & \mu_2\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}\mu\\ 1\end{bmatrix}. \tag{22}$$

we have

$$\begin{bmatrix}\mu\\ 1\end{bmatrix} = k_1\begin{bmatrix}\mu_1\\ 1\end{bmatrix}+k_2\begin{bmatrix}\mu_2\\ 1\end{bmatrix}, \tag{23}$$

so that

$$\hat{x} = k_1\hat{x}_1 + k_2\hat{x}_2.$$

Q.E.D.

Remark 3.5. The two fund theorem explores the fact that Markowitz optimal portfolio as a function of the return μ is affine. This is a structure of the solution of the entropy maximization problem when we have a quadratic function as the negative of the generalized entropy. In pointing out that all efficient Markowitz portfolios are generated by just two basic efficient portfolios, the two fund theorem greatly simplifies that task of determining Markowitz portfolios. In practice, one can often use two broad based indices to approximate the two basic efficient portfolios. This can be viewed as the theoretical foundation for the passive investment strategy of buy and hold broad based indices.

4 Capital Asset Pricing Model

Capital asset pricing model (CAPM) is a theoretical model independently proposed by Lintner [18], Mosssin [21], Sharpe [25] and Treynor [28] for pricing a risky asset according to its expected payoff and market risk, often referred to as the beta. Mathematically the core of the capital asset pricing model can be viewed as an extension of the analysis of the Markowitz portfolio theory to include a riskless bond. Thus the model is actually simpler: the function f defined below are similar to that we used in the Markowitz portfolio theory:

$$f(x) = \frac{1}{2}Var(x \cdot S_1) = \frac{1}{2}x^\top \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix} x, \tag{24}$$

and

$$A = \begin{bmatrix} \mathbf{E}(S_1) \\ S_0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{25}$$

As discussed in Remark 3.1 we directly consider portfolio $x \in \mathbf{R}^{M+1}$ and the following entropy maximization problem

$$\inf[f(x) \mid Ax = b]. \tag{26}$$

Direct calculation yields

$$f^*(y) = \begin{cases} +\infty & y_0 \neq 0 \\ \frac{1}{2}\hat{y}\Sigma^{-1}\hat{y} & \text{otherwise.} \end{cases} \tag{27}$$

Using the duality relationship in (3) we can calculate the value of the entropy maximization problem (26) to be

$$\begin{aligned} f^* \circ A^\top(z) &= \begin{cases} +\infty & z_1R + z_2 \neq 0 \\ \frac{1}{2}z^\top \hat{A}\Sigma^{-1}\hat{A}^\top z & [R, 1]z = 0 \end{cases} \\ &= \begin{cases} +\infty & z_1R + z_2 \neq 0 \\ \frac{1}{2}(\alpha - 2\beta R + \gamma R^2)z_1^2 & z_2 = -z_1R \end{cases}. \end{aligned} \tag{28}$$

Since Σ is positive definite, $z^\top \hat{A}\Sigma^{-1}\hat{A}^\top z > 0$ whenever $z \neq 0$. Thus $\Delta := \alpha - 2\beta R + \gamma R^2 > 0$. We can calculate that

$$\begin{aligned} (f^* \circ A^\top)^*(b) &= \max_z \left\{ z^\top \begin{bmatrix} \mu \\ 1 \end{bmatrix} - (f^* \circ A^\top)(z) \right\} \\ &= \max_{z_1} \left\{ z_1(\mu - R) - \frac{1}{2}z_1^2\Delta \right\} = \frac{(\mu - R)^2}{2\Delta} \end{aligned} \tag{29}$$

We know it only make sense to involve risky assets when we can expect an excess return, that is, return μ should be higher than the riskless return R. Placing the optimal portfolio again in the (σ, μ)-plane we see that they are all on the straight line

$$\sigma = \frac{\mu - R}{\sqrt{\Delta}} \text{ or } \mu = R + \sigma\sqrt{\Delta}. \tag{30}$$

Again we see the affine structure of the solution. Thus, all the optimal solution, represented in the (σ, μ)-plane, should be the convex combination of two basic optimal solutions. This is rather similar to the two fund theorem in the previous section. A convenient choice for the two basic optimal solutions are taking one portfolio that contains only the riskless bond and another portfolio with only risk asset. Clearly, the portfolio that contains only risk assets has to reside on the Markowitz efficient frontier. We call this portfolio the *market portfolio*. Summarizing we get what people often referred to as the two fund separation theorem.

Theorem 4.1. (Two Fund Separation Theorem) *All the optimal portfolios in the CAPM model are convex combinations of the riskless bond and the market portfolio.*

Now we turn to the issue of calculating the optimal portfolio. Denoting the solutions to the primal and dual problems by $\bar{x}$ and $\bar{z}$, respectively, we have

$$A^\top \bar{z} = f'(\bar{x}) = \begin{bmatrix} 0 \\ \Sigma\hat{\bar{x}} \end{bmatrix} \tag{31}$$

or

$$\hat{\bar{x}} = \Sigma^{-1}\hat{A}^\top \bar{z} \tag{32}$$

and

$$\bar{z}_1 R + \bar{z}_2 = 0 \tag{33}$$

It follows from (29) that $\bar{z}_1 = (\mu - R)/\Delta$ so that by (33) we have

$$\bar{z} = \bar{z}_1 \begin{bmatrix} 1 \\ -R \end{bmatrix} = \frac{\mu - R}{\Delta} \begin{bmatrix} 1 \\ -R \end{bmatrix}. \tag{34}$$

Combining (32) and (34) we have a clean representation of the risky part of the optimal portfolio

$$\hat{\bar{x}} = \frac{\mu - R}{\Delta}\Sigma^{-1}\hat{A}^\top \begin{bmatrix} 1 \\ -R \end{bmatrix}. \tag{35}$$

We can calculate the capital allocated to the risky part of the portfolio to be

$$1 - \bar{x}_0 = \hat{S}_0 \cdot \hat{\bar{x}} = \frac{\beta - \gamma R}{\Delta}(\mu - R). \tag{36}$$

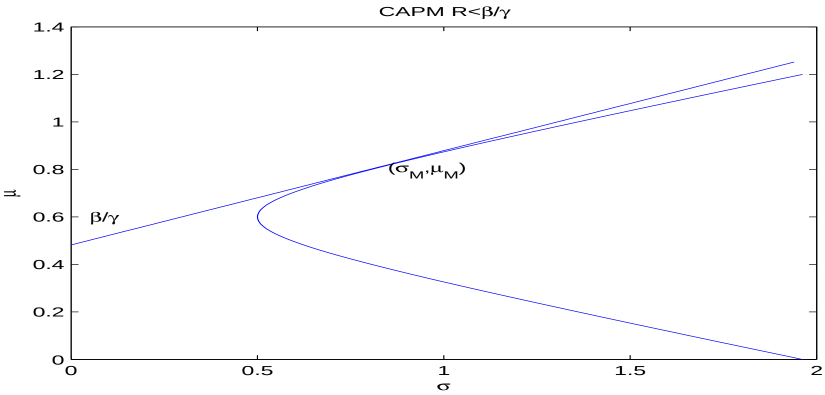

Figure 2: Capital market line

From (36) we see that to get an excess return $\mu > R$, we need to long risky assets when $R < \beta/\gamma$ and short risky assets when $R > \beta/\gamma$. When R is exactly β/γ, no portfolio can achieve excess return and there is no benefit involving risky assets in the portfolio.

Next we focus on the case when $R < \beta/\gamma$. We observe that when the right hand side of (36) is 1 we have a portfolio that is entirely consisting of risky assets. The corresponding optimal portfolio is the market portfolio

$$\bar{x}_M = \left(0, \frac{1}{\beta - \gamma R}\Sigma^{-1}\hat{A}^\top \begin{bmatrix} 1 \\ -R \end{bmatrix}\right). \tag{37}$$

The corresponding return μ_M and the standard deviation σ_M are given below

$$\mu_M = R + \frac{\Delta}{\beta - \gamma R} \tag{38}$$

$$\sigma_M = \frac{\sqrt{\Delta}}{\beta - \gamma R}. \tag{39}$$

We observe that the market portfolio is independent in μ. Moreover, as alluded to in the two fund separation theorem all the optimal portfolios are a combination of the market portfolio and the riskless bond. On the (σ, μ)-plane they are all located on the line (see Fig. 2)

$$\mu = R + \sqrt{\Delta}\sigma. \tag{40}$$

We call this line the *capital market line*

We can summarize the above as:

Theorem 4.2. (Capital Market Line) *Optimal portfolios represented as points in the (σ, μ)-plane are all located on the* capital market line

$$\mu = R + \sigma\sqrt{\Delta},$$

where $\Delta = \alpha - 2\beta R + \gamma R^2$*. The capital market line is tangent to the boundary of the Markowitz bullet at*

$$(\sigma_M, \mu_M) = \left(\frac{\sqrt{\Delta}}{\beta - \gamma R}, R + \frac{\Delta}{\beta - \gamma R}\right)$$

and intercept the μ *axis at* $(0, R)$*. The portfolio corresponding to* (σ_M, μ_M) *is*

$$\bar{x}_M = \left(0, \frac{1}{\beta - \gamma R}\Sigma^{-1}\hat{A}^\top \begin{bmatrix} 1 \\ -R \end{bmatrix}\right).$$

and is called the capital market portfolio.

Alternatively we can write the slope of the capital market line as

$$\sqrt{\Delta} = \frac{\mu_M - R}{\sigma_M}. \tag{41}$$

This quantity is called the *price of risk* and we can rewrite the equation for the capital market line as

$$\mu = R + \frac{\mu_M - R}{\sigma_M}\sigma. \tag{42}$$

Next we discuss how to use the capital market line to price a risky asset. The capital asset pricing model assumes that adding a fair priced risky asset to the market should not change the capital market line. The price is indirectly reflected in the expected return of the asset. Thus, given a risky asset a^i, we try to determine the its expected return μ_i.

Theorem 4.3. (Capital Asset Pricing Model) *Suppose that we know a financial market* S *with a riskless bond returning* R*. Let* a^i *be a fair priced risky asset with expected percentage return* μ_i*. Then*

$$\mu_i = R + \beta_i(\mu_M - R). \tag{43}$$

Here $\beta_i = \sigma_{iM}/\sigma_M^2$ *is called the beta of* a^i*, where* $\sigma_{iM} = cov(a^i, \bar{x}_M \cdot S)$ *is the covariance of* a^i *and the market portfolio.*

Proof. Consider a portfolio relies on the parameter α that consists the risky asset a^i and the market portfolio:

$$p(\alpha) = \alpha a^i + (1 - \alpha)\bar{x}_M \cdot S. \tag{44}$$

Denote the expected return and the standard variation of $p(\alpha)$ by μ_α and σ_α, respectively, we have

$$\mu_\alpha = \alpha\mu_i + (1 - \alpha)\mu_M, \tag{45}$$

and

$$\sigma_\alpha^2 = \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\mu_M^2. \tag{46}$$

The parametric curve $(\sigma_\alpha, \mu_\alpha)$ must lie below the capital market line because the latter consists of optimal portfolios. On the other hand it is clear that when $\alpha = 0$ this curve coincide with the capital market line. Thus, the capital market line is an tangent line of the parametric curve $(\sigma_\alpha, \mu_\alpha)$ at $\alpha = 0$. It follows that

$$\frac{\mu_M - R}{\sigma_M} = \left[\frac{d\mu_\alpha}{d\sigma_\alpha}\right]_{\alpha=0} = \frac{\sigma_M(\mu_i - \mu_M)}{\sigma_{iM} - \sigma_M^2}. \tag{47}$$

Solving for μ_i we derive

$$\mu_i = R + \beta_i(\mu_M - R). \tag{48}$$

Q.E.D.

5 Fundamental Theorem of Asset Pricing

In this section we consider the problem of pricing a risky asset from a different perspective based on the principle of no arbitrage. This perspective leads to the fundamental theorem of asset pricing (FTAP) a fundamental result pioneered by Cox and Ross [7] and developed in progressing generality in the past several decades by many researchers (see [8, 9, 13, 12]). FTAP links the no arbitrage principle to the existence of equivalent martingale measures which can be used to price risky assets including contingent claims in a given financial market. Our discussion starts with portfolio utility (seeing as a generalized entropy) maximization problem and then view the equivalent martingale measure (also called the risk neutral measure) as the dual solution follows the idea in [6, 29].

Gain without risk is what every investor desires. Such opportunities arguably will not last. Since when everyone tries to chase it the price will move up that will eventually eliminate the opportunity. Based on this observation, in modeling a financial market a guiding principle is that arbitrage should not exist. The following is a formal definition.

Definition 5.1. (Arbitrage) *We say that a portfolio Θ is an arbitrage if it involves no risk, $\Theta \cdot (S_1 - S_0) \geq 0$ yet has opportunity to gain something $\Theta \cdot (S_1 - S_0) \neq 0$.*

The Fundamental Theorem of Asset Pricing (FTAP) links no arbitrage with the existence of risk neutral or martingale measures defined below:

Definition 5.2. (Equivalent martingale measure) *We say that Q is an* Equivalent Martingale Measure (EMM) *on economy $(\Omega, \mathcal{F}, P)$ for financial market S provided that, for any atom B_i of $\mathcal{F}$, $Q(B_i) \neq 0$ if and only if $P(B_i) \neq 0$, and*

$$\mathbf{E}^Q[S_1] = S_0.$$

The significance of the theorem is that knowing an equivalent martingale measure Q can be used to pricing financial assets. Suppose $\phi(S_1)$ is a function of the financial assets in the market represents the payoff of a contingent claim at time $t = 1$. Then $\phi_0 = \mathbf{E}^Q[\phi(S_1)]$ is a reasonable price for this derivative at $t = 0$ in the sense that using this price will not create any arbitrage opportunities. To understand FTAP let's denote the set of gains by

$$W := \{\Theta \cdot (S_1 - S_0) : \Theta \in port[S]\} \subset RV(\Omega, \mathcal{F}, P).$$

We can see that, in fact, W is a subspace of $RV(\Omega, \mathcal{F}, P)$. It is not hard to see that if Θ is an arbitrage portfolio then $\Theta \cdot (S_1 - S_0) \in RV(\Omega, \mathcal{F}, P)^+ \backslash \{0\}$, where $RV(\Omega, \mathcal{F}, P)^+$ is the cone of nonnegative random variables. Thus, no arbitrage can be described as

$$W \cap RV(\Omega, \mathcal{F}, P)^+ \backslash \{0\} = \emptyset.$$

Traditional proofs of the FTAP rely on applying an appropriate version of the cone separation theorem to ensure that there is a hyperplane separating W and $RV(\Omega, \mathcal{F}, P)^+$. Then, a scaling of the normal vector of such a separating hyperplane gives us an equivalent martingale measure.

The fact that such an equivalent martingale measure comes from a generic separation theorem is often interpreted as the no arbitrage price being independent of investor's preferences. However, we derive FTAP using a framework of entropy maximization where the 'entropy' is a utility function that captures the risk aversion of a typical investor. This approach also shows that martingale measures are actually related to the risk aversion of investors. We consider a general extended valued upper semicontinuous utility function u that satisfies the following conditions:

(u1) (Risk aversion) u is strictly concave,

(u2) (Profit seeking) u is strictly increasing and $\lim_{t\to+\infty} u(t) = +\infty$,

(u3) (Bankruptcy forbidden) For any $t < 0$, $u(t) = -\infty$,

A rational investor with a utility function u satisfying conditions (u1)-(u3) will try to maximize the expected utility of the final wealth among all portfolios in $port[S]$. In other words, if $w_0 > 0$ is the initial wealth of the investor, he wants to solve the following portfolio utility maximization problem:

$$\sup\{\mathbf{E}[u(w_0 + \Theta \cdot (S_1 - S_0)] : \Theta \in port[S]\}. \tag{49}$$

It turns out that an arbitrage opportunity is exactly characterized by the optimal value for problem (49) to be $+\infty$.

Theorem 5.3. (Characterizing arbitrage with utility optimization) *The portfolio space $port[S]$ contains an arbitrage if and ony if the optimal value of the utility optimization problem is $+\infty$*

Proof. The "only if" part is easy: if $\Theta \in port[S]$ is an arbitrage then so is $r\Theta$ for any $r > 0$. Then it is easy to see that $\mathbf{E}[u(w_0 + r\Theta \cdot (S_1 - S_0)] \to +\infty$ as $r \to +\infty$.

To prove the "if part" assume the optimal value for problem (49) is $+\infty$. Then there exists a sequence $\Theta^n \in port[S]$ such that $\mathbf{E}[u(w_0 + \Theta^n \cdot (S_1 - S_0)] \to +\infty$ as $n \to +\infty$. Necessarily, $t_n = \|\Theta^n \cdot (S_1 - S_0)\|_{RV} \to +\infty$ as n goes to ∞. Use the definition of the portfolio norm we can show that $\|\Theta^n/t_n\|$ is unifromly bounded. Thus, without loss of generality we may assume that Θ^n/t_n converges to some $\Theta^* \in port[S]$. Note that, for any n, $\Theta^n \cdot (S_1 - S_0) \geq -w_0$ by property (u3) of the utility function. Thus, $\Theta^* \cdot (S_1 - S_0) \geq 0$. Also,

$$\|\Theta^* \cdot (S_1 - S_0)\| \geq \liminf_{n\to\infty} \|\Theta^n \cdot (S_1 - S_0)/t_n\| = 1.$$

Therefore, Θ^* is an arbitrage. Q.E.D.

Given an initial wealth $w_0 > 0$, the set of all achievable wealth outcomes at the end of the one period economy $t = 1$ using all possible portfolios is

$$w_0 + \{\Theta \cdot (S_1 - S_0) : \Theta \in port[S]\} \subset RV(\Omega, \mathcal{F}, P).$$

Theorem 5.4. (Refined fundamental theorem of asset pricing) *Let S be a financial market, let u be a utility function that satisfies properties (u1),(u2) and (u3) and let $w_0 \geq 0$ be a given initial endowment. Then the following are equivalent:*

(i) *$port[S]$ contains no arbitrage.*

(ii) *The optimal value of the portfolio utility optimization problem (49) is finite value and attained.*

(iii) *There is an equivalent S-martingale measure proportional to the subdifferential of u at the optimal solution of (49).*

Proof. We use a cyclical proof.

By Theorem 5.3, $port[S]$ contains no arbitrage if and only if the optimal value of problem (49) is finite and, therefore, (i) implies (ii).

Implication (ii) $\to$ (iii) is the key and we use entropy maximization. Observing that the utility optimization problem (49) can be written equivalently as

$$\begin{aligned} p := \max \quad & \mathbf{E}[u(y)] \\ \text{subject to} \quad & y \in w_0 + W. \end{aligned} \tag{50}$$

Alternatively we can write (50) as an entropy optimization problem

$$\begin{aligned} -p = \text{minimize} \quad & \mathbf{E}[(-u)(y)] \\ \text{subject to} \quad & y - \Theta \cdot (S_1 - S_0) = w_0. \end{aligned} \tag{51}$$

In this problem the variable $x = (y, \Theta)$, $f(x) = E[(-u)(y)]$ and the moment condition is $Ax = y - \Theta \cdot (S_1 - S_0) = w_0$. Thus, $\text{dom } f = RV(\Omega, \mathcal{F}, P)^+ \times port[S]$ and the constrain qualification condition $w_0 \in \text{ri } A \text{ dom } f$ holds. Thus, the dual problem to (51) has a solution λ which is the primal Lagrange multiplier. We have already known from the proof of the Theorem 5.4 that the primal problem has a solution (y^*, Θ^*). Then the Lagrangian

$$\begin{aligned} L((y,\Theta),\lambda) &= \mathbf{E}[(-u)(y)] + \langle \lambda, y - \Theta \cdot (S_1 - S_0) - w_0 \rangle \\ &= \mathbf{E}[(-u)(y)] + \langle \lambda, y - w_0 \rangle - \langle \lambda, \Theta \cdot (S_1 - S_0) \rangle \\ &= \mathbf{E}[(-u)(y) + \lambda(y - w_0)] - \langle \lambda, \Theta \cdot (S_1 - S_0) \rangle \end{aligned}$$

attains minimum at (y^*, Θ^*). It follows that $\langle \lambda, S_1 - S_0 \rangle = 0$ and $-\lambda(B_i) \in \partial(-u)(y^*(B_i)), i = 1, 2, \ldots, N$ for $P(B_i) > 0$. Since $-u$ is strictly decreasing we have $\lambda(B_i) > 0$ whenever $P(B_i) > 0$. Moreover, dividing $\langle \lambda, S_1 - S_0 \rangle = \mathbf{E}[\lambda(S_1 - S_0)] = 0$ by $\mathbf{E}[\lambda]$ and noticing that S_0 is a constant vector we get

$$\mathbf{E}[(\lambda/\mathbf{E}[\lambda])S_1] = S_0.$$

This is to say that $Q = (\lambda/\mathbf{E}[\lambda])P$ is a martingale measure equivalent to P. Thus, (ii) implies (iii). We can see that this martingale measure is indeed a scaling of the Lagrange multiplier.

Finally, if (iii) is true then there cannot be any arbitrage in $port[S]$ because adding an arbitrage to the optimal solution of (49) will improve it. Thus, (iii) implies (i) and we have completed a cyclic proof of the equivalence of (i), (ii) and (iii). Q.E.D.

Remark 5.5. Although no arbitrage is equivalent to the existence of an equivalent martingale measure is well known, as pointed out in [29] the proof of Theorem 5.4 using a class of utility functions says more: when the martingale measure is not unique, the dual problem actually points to one particular martingale measure. Thus, in principle, every choice of equivalent martingale measure (corresponding to a particular price of the contingent claim) can be viewed as a particular portfolio optimization problem with a corresponding concave utility function. In particular, when the market is not complete there are many possibilities in selecting the utility functions. Thus, the pricing of contingent claims do rely on the trader's preference.

6 Selecting a Pricing Martingale Measure by Entropy Maximization

We have seen in the previous section that equivalent martingale measure is related to the investor's risk aversion and, in general, not unique. Thus, for a contingent claim, its price under the no arbitrage principle with equivalent martingale measures is not unique. Question arise as to how to choose an appropriate martingale measure. Theoretically, if the investor's risk aversion, described by a utility function is specified then one can determine the martingale measure according to the refined FTAP in the previous section.

The problem with this approach is that it is well known that specify or calibrate the utility function is very difficult in practice. Moreover, even if a utility function is known, deriving a corresponding martingale measure according to FTAP needs to solve the portfolio optimization problem which is also quite difficult. On the other hand determining a equivalent martingale measure for a financial market S on a finite probability space amounts to solve a matrix equation which is easy. Therefore, in practice practitioners usually directly deal with equivalent martingale measures. When the martingale measures are not unique the question is then how to choose one that is reasonable. Using a criterion of maximizing the entropy was proposed by Stutzer [27] and Borwein, Choksi and Lamarchel in [2]. The mathematical formulation is

$$\min\{f(Q) : Q \in \mathcal{M}\}, \tag{52}$$

where $\mathcal{M}$ is the set of all martingale measures on market S and f is the negative of an entropy. Often a choice for f is the Boltzmann-Shannon entropy

$$f(x) = \sum_{n=1}^{N} p(x_n), \tag{53}$$

where

$$p(t) := \begin{cases} t \ln t - t & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ +\infty & \text{if } t < 0, \end{cases}$$

but other entropy functions can also be used. Selecting Boltzmann-Shannon entropy means assuming no prior knowledge on investor's view on the probability distribution.

On the other hand, considering $f + \iota_{\mathbf{R}^N_+}$ if necessary, we can assume that $\operatorname{dom} f \subset \mathbf{R}^N_+$. Then we can rewrite problem (52) as

$$\min\{f(Q) : \langle Q, S_1 - S_0 \rangle = 0, \langle Q, 1 \rangle = 1\}. \tag{54}$$

Let (Θ, w) be the dual variable in which Θ and w are Lagrange multipliers corresponding to the constraints $\langle Q, S_1 - S_0 \rangle = 0$ and $\langle Q, 1 \rangle = 1$, respectively. Then the dual problem of (54) is

$$\max\{w - f^*(w + \Theta \cdot (S_1 - S_0))\}. \tag{55}$$

We can view (55) as a portfolio utility maximization problem where w plays the role of initial endowment. Thus, we can see that selecting a pricing martingale measure by maximizing an entropy eventually is still implicitly related to a utility maximization problem.

7 Super/sub hedging bounds

As alluded to before, when the martingale measures are not unique, for a given contingent claim we can derive multiple prices of the contingent claim from those martingale measures that are consistent with the no arbitrage principle. Taking sup and inf of these prices we derive a range outside of which arbitrage opportunity emerges. Duality helps to unveil how to construct a portfolio to take such advantages. We will analyze the case when market price exceeds the sup. This will produce an opportunity for super hedging. The discussion about the situation when price falls below the inf is similar.

Let $\phi(S_1)$ be the payoff of the contingent claim at $t = 1$. Define

$$U = \max\{\mathbf{E}^Q[\phi(S_1)] \mid Q \in \mathcal{M}\}. \tag{56}$$

Then U is the upper bound of no arbitrage pricing, called the super hedging bound. Defining

$$f(Q) = \mathbf{E}^Q[-\phi(S_1)] + \iota_{\mathbf{R}^N_+}(Q), \tag{57}$$

we can represent U as the negative value of an entropy maximization problem

$$U = -\min_Q\{f(Q) : \langle Q, S_1 - S_0\rangle = 0, \langle Q, 1\rangle = 1\}. \tag{58}$$

Using the strong duality (54) and (55) we have

$$U = -\max_{w,\Theta}\{w - f^*(w + \Theta \cdot (S_1 - S_0))\}. \tag{59}$$

We can directly calculate that

$$f^*(y) = \iota_{[z:z\leq -\phi(S_1)]}(y). \tag{60}$$

Thus,

$$\begin{aligned} U &= \min_{w,\Theta}\{-w + f^*(w + \Theta \cdot (S_1 - S_0))\} \qquad (61)\\ &= \min_{w,\Theta}\{-w \mid w \leq -\Theta \cdot (S_1 - S_0)) - \phi(S_1)\} \end{aligned}$$

The dual representation in (61) provides us a way of finding the super-hedging portfolio to take advantage the arbitrage opportunity should the market price of the contingent claim exceeds the supper-hedging bound U. In fact, the second line in (61) tells us that we can derive the super-hedging portfolio by solving the linear programming problem

$$\min_{w,\Theta}\{-w \mid w + \Theta \cdot (S_1 - S_0)) \leq -\phi(S_1)\}. \tag{62}$$

It is easy to see that one can rewrite (62) as

$$U = \min_{\Theta}\sup_{\omega\in\Omega}[\Theta \cdot (S_1 - S_0))(\omega) + \phi(S_1)(\omega)]. \tag{63}$$

This is the formula derived by Kahalé in [16] using a separation theorem argument.

8 Conclusion

Markowitz portfolio theory, the capital market pricing model, fundamental theorems of asset pricing, selecting equivalent pricing martingale measures using entropy maximizations and finding super/sub-hedging bounds and portfolios are several important results in financial economies. We illustrated that they can all be understood in the framework of entropy maximization. So many important results in financial economics involving this fundamental principle in physics demonstrated the heavy influence of methods in physical sciences to financial research.

This link to physical science brings about welcome rigor and quantitative precision into financial research. On the other hand, the principle of entropy maximization is proposed in statistical mechanics. Statistic mechanics deals with complex systems consist of many identical microscopic elements. The impact of each of the elements to the system as a whole is negligible. While these models resembles Aumann's idealized atom less economy, they are significantly different from the real financial market. Two of the main differences are: first agents in a financial market are not uniform in their sizes and impacts to the market as a whole. Many big financial institutes can swing the market in a significant way. In particular, in a crises the failure of any of those big players can cause turmoil to the whole market as the 2008 financial crises and many of its predecessors have shown. The second main difference is that agents in a financial market are humans. Instead following a fixed physical law they interact with each other and their behaviors are also determined by human psychology. For these reasons those fundamental results in finance derived above using the entropy maximization methods should be treated as a rough sketch of a road map that needs to be used with caution in practice.

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