Investment system specific European option pricing intervals

Q. J. Zhu
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA
e-mail: zhu@wmich.edu

August 22, 2006

Abstract

Consider an investment system with a nonnegative expected return in a one period economy. We show that, for an option with a given strike price, there exists a pricing interval $[p_C, p_W]$ such that replacing the original investment with the option will benefit judging by the Kelly criterion only when the price of the option lies outside of the interval. More specifically, buying call options with a price less than $p_C$ or writing covered call options with a price greater than $p_W$ will improve the investment system. Bounds for these thresholds are established. This investment system specific option pricing interval is compatible with option valuation methods of using a replicating portfolio or a risk neutrality argument.

Key Words. Option pricing interval, incomplete market, investment systems, the Kelly criterion, efficiency index, convex optimization.

Acknowledgement. I thank Professor D. Madan for bringing my attention to recent related research and for his helpful comments on an earlier version of the paper.
1 Introduction

Option valuation draws increasing attention due to the prosperity of financial option products and other forms of contingent claims in the financial industry that can be represented using options. One of the most influential paradigm in this area is the method of valuing options by self-financing dynamic hedging proposed in the pioneering work of Black and Scholes [Black and Scholes, 1973] and Merton [Merton, 1973]. The idea is to trade continuously to maintain a self-financing portfolio of the underlying investment instrument and a bond generating the market risk-free rate in such a way that the payoff of the portfolio replicates that of the option. Under the no arbitrage principle, the cost of the portfolio will give us the fair price of the option. While this method and its extensions and refinements over the past several decades have provided important guidance to the pricing of options, a major concern has been that continuous trading cannot be implemented due to the trading cost. Thus, when the price of the option deviates from the predicted value creating a theoretical arbitrage opportunity, how to take advantage of it remains elusive.

In this paper we approach the problem from a practitioner’s point of view. Suppose an investor has a private profitable investment system for optionable investment instruments in a one period economy. We ask the question, for an option with a fixed strike price, what are the valuations of the option at which one can use it to replace the original investment and thereby improve it judging by the Kelly criterion? The assumptions of a one period economy and of allowing for only replacing the original investment with an option are both aimed at limiting trading. Under these assumptions, for each investment opportunity, only one trade is allowed which takes the other extreme in comparison to the continuous trading scheme.

To explain our findings, let us focus on an investment system for optionable stocks. We assume that the investment system contains one (representative) stock whose spot price at the beginning of the period is standardized to 1 and the price of the stock at the end of the period is a random variable $X$ in terms of the real value. The investor knows the subjective probability distribution of $X$, and the investment system is profitable so that $E[X - 1] \geq 0$. In this case, only a call option could possibly improve the investment system. For simplicity, we illustrate using the at money call. To trade a call option, there are two sides: the buyer of the call and the writer of the covered call. We can view these two sides as engaged in a zero sum game relative
to the underlying stock. Thus, heuristically, if one side is doing better than
the stock, then that is probably at the expense of the other side due to a
favorable option price. This proves to be true. In addition we find that, in
general, the option trading is not efficient in the sense that there exists a
pricing interval inside which both the buyer and the writer of the option un-
der perform the stock. More precisely, there exists three critical call option
premiums: $p^C \leq p^* \leq p^W$ having the following properties. When the option
premium $p < p^C$, buying call options will improve the investment system
and when $p > p^W$, writing covered call options will improve the investment
system. The $p^*$ gives us an option premium level at which buying calls and
writing covered calls are the same, and they are no better than the original
investment system. In fact, the latter is true for any option premium $p$
that belongs to the option pricing interval $[p^C, p^W]$. Moreover, we establish the
following bounds for these three critical option premiums

$$\mathcal{E}[(X - 1)^-] \leq p^C \leq p^* \leq p^W \leq \bar{p} \leq \mathcal{E}[(X - 1)^+] \quad (1.1)$$

Here $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, and $\bar{p} = \mathcal{E}[(X - 1)^+] / \mathcal{E}[X]$, which is
the option premium that balances the expected percentage returns of buying
calls and writing covered calls. This common expected percentage return
also coincides with $\mathcal{E}[X - 1]$, the expected percentage return of the original
investment system. The general pattern in (1.1) also holds for in the money
and out of the money call options.

In a complete market two main methods of pricing options are by using
a replicating portfolio pioneered in [Black and Scholes, 1973, Merton, 1973]
and the risk neutrality argument proposed in [Cox and Ross, 1976]. We show
that our method is compatible to both in the sense that if the option payoff
can be replicated, then $p^C = p^* = p^W$ and the common value coincide with
the option valuation derived by the aforementioned classical methods.

Option pricing intervals in incomplete markets have been studied us-
ing other methods. For example, Levy [Levy, 1985] adopted a stochastic
dominance approach, refining the earlier work of Merton [Merton, 1973];
Ledoit [Ledoit, 1995] and Bernardo and Ledoit [Bernardo and Ledoit, 2000]
relied on the Sharpe ratio as the comparing criterion; and Cochrane and
Saá Requejo [Cochrane and Saá Requejo, 2000] used discount factors moti-
vated by the Sharpe ratio. Those pricing intervals are not necessarily
compatible to option valuations derived by replicating portfolio or the risk
neutral argument when they are applicable. Moreover, Jarrow and Madan
pointed out in [Jarrow and Madan, 1997] that selecting investment opportunities based on the mean-variance preferences may lead to the rejection of free calls. This highlights the advantage of the Kelly criterion. The Kelly criterion appeared in [Kelly, 1956]. The original form is a money management scheme for gambling problems and serves as a concrete example illustrating the meaning of Shannon’s information rate in the information theory [Shannon and Weaver, 1949]. It has been developed in [Thorpe, 1962, Thorp, 1997, Vince, 1990, Zhu, 2006]. In a sense, the Kelly criterion measures the useful investment information in an investment system adjusted for the ‘noise’ in the form of volatility. The information theoretical approach has been shown to be a useful tool in analyzing financial problems (see Thorp [Thorp, 1971], Erkip and Cover [Erkip and Cover, 1998] and Cover [Cover, 1998] and the references therein). Discussions in the current paper provide further indication of the relevance of this method.

A different approach was taken by Carr, Geman and Madan in [Carr, Geman, and Madan, 2001]. A set of probability measures were used to represent the beliefs of multiple market participants, and an acceptable price for an option is determined by its acceptability to all the players. The study of portfolio optimization involving options is also tangential to our analysis. One can determine whether a price of the option is favorable by whether at that price the option is included in the optimal portfolio. Recent study by Carr and Madan [Carr and Madan, 2001] and by Iihan, Jonsson and Sircar [Iihan, Jonsson, and Sircar, 2005] are examples of research in this direction and both discuss one period models. Assuming a continuum of option strikes and general increasing concave utilities Carr and Madan discussed both partial and general equilibria in [Carr and Madan, 2001]. In contrast, Iihan, Jonsson, and Sircar assumed continuous trading in stocks and a finite number of statical option positions using the exponential utility function in [Iihan, Jonsson, and Sircar, 2005]. Besides the different choice of criteria for comparing investment systems, our study differs from those mentioned above in two other aspects. First, we take the perspective of an investor and only assume knowledge of the probability distribution of the payoff of the investor’s own investment system. Second we assume high barrels of entering an investment position. This is due to operational considerations. Given the track record of an investment system, one can numerically compute option premium thresholds $p_C$ and $p_W$ for a finite number of strike prices as a guide to when an investment could be replaced by a corresponding option favorably. To do the same for optimal portfolios involves many more scenarios. Imple-
mentation of an optimal portfolio is more costly in terms of trading cost and even more so in terms of operation time. In addition, optimal portfolios are known to be sensitive to the inputs. Thus, simple thresholds for triggering a favorable option replacement investment strategy are more practical. On the other hand, the general equilibrium problem regarding to who will take the other side of the trade is not explicitly discussed in this paper.

The remainder of the paper is arranged as follows. In the next section we discuss the Kelly’s criterion and preliminary results related to its properties; in Section 3 we explore relationship between investment systems and their related option investment strategies using concrete examples to gain some insights; we then prove that what we observed is true in general in Section 4; in Section 5 we discuss the relationship of our results with methods for pricing options by replicating portfolio and by using a risk neutrality argument; concluding remarks and considerations of how to apply the results in investment decisions are contained in Section 6.

2 Investment systems and Kelly’s criterion

We use a trade as the basic unit of an investment system. A trade is the process of acquiring an investment instrument using the investment capital and subsequently liquidating it and returning the proceeds to the investment capital. We define an investment system I as a set of trades. In other words, an investment system is a systematical way to execute trades. We will model this as a one period economy containing a risky investment instrument, say, a (generic) stock to be concrete, and a risk-less bond. We assume that all trades in this economy are initiated at time 0 and cleared at time 1. We further assume that accounting is in terms of real value discounted for the rate of return on risk-less assets so that the rate of return on the risk-less bond is 0. We standardize the spot price of the stock at time 0 to be 1. Then an investment system I is characterized by the probability distribution of the percentage gain of the trades at time 1. We will characterize this probability distribution by the profile

\[ \{g_n : n = 1 \ldots N \}, \]

of I where \( g_n \) are the percentage gains (losses) of historical trades of the investment system in a relevant period. This amounts to back test an investment system. Since only the profile of an investment system will be used in
our analysis, we will not distinguish an investment system and its profile in
the sequel and will use notation

\[ I = \{ g_n : n = 1 \ldots N \} . \] (2.1)

In short, we model the behaviors of an investment system \( I \) of trading \( N \)
times in a relevant period by a one period economy containing a single risky
investment instrument (called stock for simplicity) whose percentage payoff
has a probability distribution characterized by (2.1). In this paper, we will
often discuss directly in terms of the investment system \( I \) represented by (2.1)
so that the notation and formulae can be used directly when the statistical
data of an investment system is given.

The phrase “relevant period” is vague and needs some explanation. What
we do here is to try to use past experience to guide the future. One strategy
could be to use all the data available. Then, the relevant period would be
the entire time that the investment system has been used so far. However,
we may not be able to get all the data. Even if we can, data from way in
the past may not be that relevant to the current market conditions. Also,
if we believe that we are facing a bull market ahead then the records of the
investment system in a recent bull market might be more relevant. Thus, in
practice, the relevant period is simply chosen as a period that, we believe,
bears most significance. Finally, we note that while the choice of the relevant
period for collecting data is pertinent to the investment practice, it has no
influence upon the theoretical discussion.

Next we consider replacing trades in an investment system by correspond-
ing options in this one period economy. This means that we assume the
maturity of these options is also at time 1. We assume that the investment
system is at least not losing money, that is, \( \mathcal{E}[I] \geq 0 \) so that we need only
to consider call options. Again we standardize the spot price of the stock
to 1 at time 0. We indicate the difference between the spot price and the
strike price of the option by a percentage \( a \) in the money relative to the spot
price so that the strike price in terms of the percentage of the spot price is
\( 1 - a \). We consider \( a < 1 \) since \( a \geq 1 \) represents a strike equal to or less than
0. By allowing \( a \) to take all the values less than 1 we can handle options
in the money, at the money and out of the money in a unified framework.
Clearly \( a = 0 \) corresponds to at the money options. When \( a < 0 \) the call
option is in fact out of the money. For example suppose the spot price of a
stock is $80. Consider two call options with strike prices of $60 and $100,
respectively. Then the first one is \((80 - 60)/80 = 25\%\) in the money and second is \((80 - 100)/80 = -25\%\) in the money (or 25\% out of the money). We use \(p\) to denote the percentage premium of the call option relative to the spot price. We will assume \(p > a^+\) so that there is no arbitrage opportunity. Given an investment system \(I\), replacing a trade in \(I\) with a corresponding trade of buying or writing a call option yields a different investment system. To characterize these related option investment systems it is useful to consider the following operations on an investment system \(I\). First we define a shift and a scale of an investment system by \(r\) as

\[ I + r := \{g_n + r, n = 1, \ldots, N\}, \]

and

\[ rI := \{rg_n, n = 1, \ldots, N\}. \]

We will also need

\[ I^+ := \{g^+_n, n = 1, \ldots, N\}, \]

\[ I^- := \{g^-_n, n = 1, \ldots, N\}, \]

\[ M^+_I := \max\{g^+_n, n = 1, \ldots, N\} \text{ and } M^-_I := \max\{g^-_n, n = 1, \ldots, N\}. \]

Denoting by \(C_I(a, p)\) and \(W_I(a, p)\) the corresponding option investment systems of replacing each trade in \(I\) by buying a call option and by writing a covered call option \(a\) in the money with a premium \(p\), respectively, we have

\[ C_I(a, p) = \frac{(I + a)^+ - p}{p} \]

and

\[ W_I(a, p) = \frac{p - a - (I + a)^-}{1 - p}. \]

Our problem becomes one of comparing investment systems \(C_I(a, p)\) and \(W_I(a, p)\) with \(I\). We will use the Kelly criterion. To motivate the definitions we start with an example.

**Example 2.1.** Let’s look at the performances of three mutual funds during a good year followed by a bad year in terms of percentage gains given in Table 2.1. Which one is better? Let us test with an $100 starting capital. Simply buying these funds and sitting until the end of the second year yields the results in Column 2 of Table 2.2. We find that the best is Fund 2 and the worst is Fund 1. However, buy-and-hold is not the only strategy we can
<table>
<thead>
<tr>
<th>Funds</th>
<th>%good year</th>
<th>%bad year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>-50%</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
<td>-18%</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
<td>-5%</td>
</tr>
</tbody>
</table>

Table 2.1: Performances of three funds.

<table>
<thead>
<tr>
<th>%invested</th>
<th>100%</th>
<th>50%</th>
<th>200%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund1</td>
<td>1000</td>
<td>1125</td>
<td>0</td>
</tr>
<tr>
<td>Fund2</td>
<td>1066</td>
<td>1046.5</td>
<td>1024</td>
</tr>
<tr>
<td>Fund3</td>
<td>1054.5</td>
<td>1028.6</td>
<td>1098</td>
</tr>
</tbody>
</table>

Table 2.2: Performances of three funds under different investment sizes.

In general, we denote the percentage investment size of the available capital by $s$, the gain in the good year by $g$, and the loss in the bad year by $l$. Then the average return per year $R(s)$ is

$$R(s) = (1 + sg)^{1/2}(1 + sl)^{1/2}$$

It is more convenient to use the return in log scale:

$$f(s) = \ln R(s) = \frac{1}{2} \ln(1 + sg) + \frac{1}{2} \ln(1 + sl).$$  \hspace{1cm} (2.2)

Drawing the log return functions for Fund 1, Fund 2 and Fund 3 together in Figure 1 we see that Fund 3 has the best potential and Fund 1 is the choice when we cannot borrow money for investing.

We now turn to the general case and our main references are [Kelly, 1956, Thorp, 1997, Vince, 1990, Zhu, 2006]. Let $I = \{g_n, n = 1, \ldots, N\}$ be an
investment system. Example 2.1 shows that, when we use the investment system \( I \) repeatedly, in addition to the profile, the performance will also depend on the size of the investment. Let us use \( s \) to denote the size of each trade as the percentage of the available capital. Using \( R_I(s) \) to denote the average exponential rate of growth of the investment capital per trade for investment system \( I \), we have

\[
R_I(s) = \Pi_{n=1}^{N} (1 + s g_n)
\]

(2.3)

As before, for ease of analysis, we will use its natural log

\[
f_I(s) = \ln R_I(s) = \frac{1}{N} \sum_{n=1}^{N} \ln(1 + s g_n)
\]

and call \( f_I(s) \) the log return function of the investment system \( I \). Normally, as a percentage of the available capital, the range of \( s \) should be \([0, 1]\). However, to explore the full potential of the investment system, we will allow \( s \) to take all the values in \((-1/M_I^+, 1/M_I^-)\) (with the convention \(1/0 = \infty\)), the domain of the log return function. For \( s \) outside \([0, 1]\), the interpretation is that \( s > 1 \) represents trade on margins and \( s < 0 \) represents trade in the opposite direction (both with margin rate 0). The maximum of \( R_I(s) \) will give us a good indication of the potential profitability of the investment system. Since the natural log is an increasing function, the maximum of \( f_I(s) \) is an equivalent indication. We call this number the efficiency index of investment system \( I \) following [Zhu, 2006] and denote it by \( \gamma(I) \), that is,

\[
\gamma(I) = \sup_{s \in (-1/M_I^+, 1/M_I^-)} f_I(s).
\]

(2.4)
Since $f_I(0) = 0$ (if one does not invest the return is 0), the efficiency index $\gamma(I)$ is always nonnegative. By definition, $\exp(\gamma(I))$ is the average expected exponential growth rate per trade for investment system $I$ under the best investment size. Thus, the larger the $\gamma(I)$, the better. We will use the efficiency index to compare different investment systems and this is the Kelly criterion.

The remainder of this section is devoted to a brief discussion of some important properties related to the Kelly criterion that will be used later. Note that if $M_I^- = 0$ ($M_I^+ = 0$), then $g_n \geq 0$ ($g_n \leq 0$) for all $n = 1, \ldots, N$. This corresponds to the case when the outcomes of the trades in $I$ are all gains (losses) and, of course, under our idealized environment of a zero margin interest rate, one should long (short) on margins as much as one can. This leads to an unlimited average rate of return per trade and, therefore, $\gamma(I) = +\infty$. In other words, an arbitrage investment system $I$ is indicated by $\gamma(I) = +\infty$. For us the interesting case is when both $M_I^-$ and $M_I^+$ are positive. It is clear from the property of the natural log function that $f_I$ is a strictly concave function and it approaches $-\infty$ as $s$ approaches the end points of the interval $(-1/M_I^+, 1/M_I^-)$. Thus, there exists a unique best investment size $\bar{s}$ such that $\gamma(I) = f_I(\bar{s})$. Since for any $r \neq 0$, $f_I(s) = f_{rI}(s/r)$, we have

$$\gamma(I) = \gamma(rI), \quad \forall r \neq 0. \quad (2.5)$$

Now we consider the other extreme when $\gamma(I) = 0$. In this case, the log return function $f_I(s) \leq 0$ for any $s$ so that $\bar{s} = 0$ is the unique best investment size. That is to say the investment system $I$ can only lose money no matter what investment size is used. We call this kind of investment system invalid. Using the condition that $f_I$ attains a unique maximum at $\bar{s} = 0$ we can characterize invalid investment systems by $f_I'(0) = 0$. We express this as:

**Theorem 2.2.** [Zhu, 2006, Theorem 5.1] An investment system $I$ is invalid if and only if

$$\mathcal{E}[I] = 0.$$  

**Proof.** We know that an invalid investment system $I$ is characterized by $f_I'(0) = 0$. It remains to observe that

$$f_I'(0) = \frac{1}{N} \sum_{n=1}^{N} g_n = \mathcal{E}[I].$$

Q.E.D.
The best investment size $\bar{s}$ can take either positive or negative values. When $\bar{s} < 0$ the investment system has to be used in the opposite direction to make money. Using the concavity of $f_I$, it is shown in [Zhu, 2006] that the sign of $\bar{s}$ is always the same as that of $E[I]$. We state this result as a lemma.

**Lemma 2.3.** [Zhu, 2006, Proposition 2.2] The sign of the best investment size $\bar{s}$ of an investment system $I$ is always the same as that of $E[I]$.

In general explicit formulae for $\bar{s}$ and $\gamma(I)$ are not to be expected (see [Zhu, 2006]). However, we can derive them for some special cases. We will need the following result, which is a special case of [Zhu, 2006, Theorem 3.2]. We include the short proof here for completeness.

**Lemma 2.4.** Let $I$ be an investment system containing $N$ trades with only three possible outcomes $\{g, 0, l\}$, where $g > 0 > l$. Suppose that the frequencies of the percentage gain of a trade to be $g, 0$ and $l$ are $\phi_g$, $\phi_0$ and $\phi_l$, respectively. Then the best investment size and the efficiency index are given by

\[
\bar{s} = -\frac{\phi_g g + \phi_l l}{(\phi_g + \phi_l)gl} \quad (2.6)
\]

and

\[
\gamma(I) = \phi_g \ln \frac{\phi_g (l - g)}{(\phi_g + \phi_l)l} + \phi_l \ln \frac{\phi_l (g - l)}{(\phi_g + \phi_l)g}. \quad (2.7)
\]

**Proof.** The log return function of $I$ is $f_I(s) = \phi_g \ln(1 + sg) + \phi_l \ln(1 + sl)$. Since the best investment size $\bar{s}$ is the unique maximizer of $f_I(s)$, it is the solution of the equation

\[
0 = f_I'(s) = \frac{\phi_g}{1 + s\phi_g} + \frac{\phi_l}{1 + s\phi_l}.
\]

Solving this equation produces formula (2.6). Then we can derive the efficiency index by $\gamma(I) = f_I(\bar{s})$. Q.E.D.

The log return function $f_I(s)$ contains much more information on the investment system $I$. For example, we have already seen that

\[
f_I'(0) = \frac{1}{N} \sum_{n=1}^{N} g_n = E(I),
\]

11
and we can represent the variation of $I$ by

$$\text{Var}(I) = -f''_I(0) - (f'_I(0))^2.$$ 

Thus, it is not surprising that the efficiency index provides a more informative assessment of investment systems. To further understand the nature of the efficiency index, let us observe its relationship with Shannon’s information rate. In Lemma 2.4, let $\phi_0 = 0$ and $g = -l$. Then our investment system $I$ becomes a game with symmetric payoffs. Kelly analyzed such system in [Kelly, 1956] and observed that in this case

$$\gamma(I) = \phi_g \ln 2\phi_g + \phi_l \ln 2\phi_l$$

is in essence Shannon’s information rate for measuring true information contents in a communication channel with noise. Kelly’s work was developed in [Thorp, 1997, Vince, 1990, Zhu, 2006]. Similar to Shannon’s information rate, the efficiency index measures useful investment information of an investment system adjusted for the ‘noise’ in the form of volatility. This provides an intuitive explanation for the relationship (2.5). Let $I$ be an investment system. Then, for $r \neq 0$, $rI$ scales the expected return of $I$ and its volatility in the same fashion. Thus, judging by the Kelly criterion $I$ and $rI$ are the same.

For investment systems with the same expected return, the less the volatility, the better. One of the quantitative characterization of this statement that we are going to use is the next lemma. Heuristically, it says calming an investment system by symmetrically moving two gains closer will make it better.

**Lemma 2.5.** Let $I = \{g_n, n = 1, \ldots, N\}$ be an investment system. For any $g_m > g_k$ and $h \in (0, (g_m - g_k)/2)$, construct a new investment system $I' = \{g_1, \ldots, g_{m-1}, g_m - h, g_{m+1}, \ldots, g_k - h, g_{k+1}, \ldots, g_N\}$. Then, for any $s$,

$$f_I(s) \leq f_{I'}(s),$$

and, therefore,

$$\gamma(I) \leq \gamma(I').$$
Proof. The result is evident from the following inequality:

\[
\begin{align*}
\frac{f_I(s) - f_I(s)}{N} &= \frac{1}{N} \left( \ln(1 + s(g_k + h)) - \ln(1 + s(g_k)) + \ln(1 + s(g_m - h)) - \ln(1 + s(g_m)) \right) \\
&= \frac{1}{N} \int_0^h \left( \frac{s}{1 + s(g_k + t)} - \frac{s}{1 + s(g_m - t)} \right) dt \\
&= \frac{1}{N} \int_0^h \frac{s^2(g_m - g_k - 2t)}{(1 + s(g_k + t))(1 + s(g_m - t))} dt \geq 0.
\end{align*}
\]

Q.E.D.

3 Examples

Let us consider an investment system \( T \) whose only possible percentage gains are \( \{g, 0, l\} \) with \( g > 0 > l \). This is the situation when we can explicitly calculate the efficiency index and we will use it to explore examples to gain some insights. Suppose that the frequency of the percentage gain of a trade to be \( g, 0 \) and \( l \) are \( \phi_g, \phi_0 \) and \( \phi_l \), respectively. Then by Lemma 2.4 we have

\[
\gamma(T) = \phi_g \ln \left( \frac{\phi_g(l - g)}{(\phi_g + \phi_l)l} \right) + \phi_l \ln \left( \frac{\phi_l(g - l)}{(\phi_g + \phi_l)g} \right). \tag{3.1}
\]

We use \( C(p) = C_T(0, p) \) and \( W(p) = W_T(0, p) \) to denote the corresponding investment systems of replacing each trade in \( T \) by buying an at money call option and by writing an at money covered call option with premium \( p \), respectively. Then the trades in \( C(p) \) have only two possible outcomes \( g/p - 1 \) and \( -1 \) with frequency \( \phi_g \) and \( 1 - \phi_g \), respectively, and the trades in \( W(p) \) also have only two possible outcomes \( p/(p - 1) \) and \( (p + l)/(1 - p) \) with frequency \( 1 - \phi_l \) and \( \phi_l \), respectively. Thus, again we can use Lemma 2.4 to explicitly calculate the efficiency index of \( C(p) \) and \( W(p) \) as functions of the option premium \( p \) given below:

\[
\gamma(C(p)) = \phi_g \ln \left( \frac{\phi_g}{p} \right) + (1 - \phi_g) \ln \left( \frac{1 - \phi_g}{g - p} \right)
\]

and

\[
\gamma(W(p)) = (1 - \phi_l) \ln \left( \frac{1 - \phi_l}{p + l} \right) + \phi_l \ln \left( \frac{-\phi_l}{p} \right).
\]

Let us examine a concrete example.
Example 3.1. Consider the investment system
\[ T = \{0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, -1\} \].

We can explicitly calculate that
\[
\gamma(T) = 0.12, \\
\gamma(C(p)) = 0.8 \ln \frac{0.4}{p} + 0.2 \ln \frac{0.1}{0.5-p},
\]
and
\[
\gamma(W(p)) = 0.9 \ln \frac{-0.9}{p-1} + 0.1 \ln \frac{0.1}{p}.
\]

Graphing them together we get Figure 2.

We observe that \( \gamma(C(0.4)) = 0 \) and \( \gamma(W(0.1)) = 0 \) gives the values 0.4 and 0.1 where the investment system of buying calls and writing covered calls are invalid, respectively. We further observe that 0.1 coincides with the average loss
\[
\mathcal{E}[T^-] = \frac{1}{10} = 0.1
\]
and that 0.4 coincides with the average gain
\[
\mathcal{E}[T^+] = \frac{8 \cdot 0.5}{10} = 0.4
\]
The graphs of \( \gamma(C(p)) \) and \( \gamma(W(p)) \) intersect in between \([0, 0.4]\) at \( p^* = 0.291 \). This is the point at which the two investment systems are equally good measured by the efficiency index. However, the common efficiency index is roughly \( \gamma(C(p^*)) = \gamma(W(p^*)) = 0.108 \), which is smaller than \( \gamma(I) = 0.12 \), indicating that some useful investment information has been lost due to the option trading. In fact, this is true for any option premium \( p \) in between \( p^C = 0.284 \) and \( p^W = 0.304 \) where the graphs of \( \gamma(C(p)) \) and \( \gamma(W(p)) \) intersect the horizontal line \( \gamma = \gamma(I) \), respectively. It is clear from the picture that when \( p < p^C \) (or \( p > p^W \)), \( C(p) \) (or \( W(p) \)) improves \( T \) and when \( p \) belongs to the interval \([p^C, p^W]\) the original investment system \( T \) is the best. Another interesting option premium not showing in the picture is the \( \bar{p} \) discussed in the introduction which equalizes the expected returns of all the three investment systems, i.e., \( \bar{p} \) satisfies
\[
\mathcal{E}[I] = \mathcal{E}[C(\bar{p})] = \mathcal{E}[W(\bar{p})].
\]
In this example \( \bar{p} = 0.308 > 0.304 = p^W \). We see that \( \bar{p} \) favors the investment system of writing covered calls. When \( p < 0.1 = \mathcal{E}[T^-] \), we have \( \mathcal{E}[W(p)] < 0 \) and \( W(p) \) loses money. The graph of \( \gamma(W(p)) \) for \( p \in (0, 0.1) \) represents the efficiency index of using the opposite of \( W(p) \), that is, buying the call option while shorting the underlying equity. Since the original investment system \( T \) makes money, shorting the underlying equity part loses money. This makes using the opposite of \( W(p) \) inferior to \( C(p) \), which explains the fact that the graph of \( \gamma(W(p)) \) is below that of \( \gamma(C(p)) \) for \( p \in (0, \mathcal{E}[T^-]) \). Similarly, the graph of \( \gamma(C(p)) \) for \( p > 0.4 = \mathcal{E}[T^+] \) represents the efficiency index of using the opposite of \( C(p) \), that is shorting the call option. Since all the entries in \( C(0.5) \) are negative, we know that \( \gamma(C(p)) \) approaches \( +\infty \) as \( p \) approaches 0.5. A similar argument shows that \( \gamma(W(p)) \) approaches \( +\infty \) as \( p \) approaches 1. This explains the intersection of the graphs of \( \gamma(C(p)) \) and \( \gamma(W(p)) \) near \( p = 0.492 \).

In the above example \( p^C < p^W \) and both \( C(p) \) and \( W(p) \) are inferior to \( I \) when \( p \) belongs to the interval \((p^C, p^W)\). This indicates the loss of some edge of the original investment system. Are there investment systems for which this does not happen? The following is an example.
Example 3.2. Consider investment system \( B = \{g, l\} \) with \( g > 0 > l \). Then \( C_B(0, p) = \{g/p - 1, -1\} \) and \( W_B(0, p) = \{p/(1-p), (p + l)/(1-p)\} \). Using Lemma 2.4 we can explicitly calculate that

\[
\gamma(I) = \frac{1}{2} \ln \frac{-(g-l)^2}{4gl},
\]

\[
\gamma(C_B(0, p)) = \frac{1}{2} \ln \frac{g^2}{4p(g-p)},
\]

and

\[
\gamma(W_B(0, p)) = \frac{1}{2} \ln \frac{-l^2}{4p(p+l)}.
\]

Solving

\[
\gamma(C_B(0, p^*)) = \gamma(W_B(0, p^*))
\]

we have

\[
p^* = \frac{gl}{l - g}.
\]

It is easy to check that

\[
\gamma(C_B(0, p^*)) = \gamma(W_B(0, p^*)) = \gamma(I).
\]

Thus, in this example \( p^C = p^W = p^* \). We can calculate that the \( \bar{p} \) satisfying

\[
\mathcal{E}[I] = \mathcal{E}[C_B(0, \bar{p})] = \mathcal{E}[W_B(0, \bar{p})]
\]

in this case is \( \bar{p} = g/(2 + g + l) \). Again, we observe that

\[
p^W = p^* = \frac{gl}{l - g} = \frac{gl}{2l - g - l} = \frac{l}{2 + \frac{2l}{g-l}} < \frac{g}{2 + g + l} = \bar{p}.
\]

Note that for the investment system in Example 3.2, our one period economy is, in fact, complete and, therefore, there exists a replication of the call option with a portfolio involving the stock and the risk-less bond. We leave it to the reader to verify that pricing the option with this replicating portfolio will yield a price that equals to \( p^C = p^W = p^* \) in Example 3.2. This is, in fact, entirely general and will be discussed further in Section 5.

We end this section with a concrete example illustrating how \( \bar{p} \) will, in general, favor writing covered call options.
Table 3.1: Percentage gains and losses of an investment system.

<table>
<thead>
<tr>
<th>Trade</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Gain</td>
<td>-1</td>
<td>-5</td>
<td>10</td>
<td>-11</td>
<td>-1</td>
<td>-9</td>
<td>30</td>
<td>20</td>
<td>-7</td>
<td>1</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

Example 3.3. Consider an investment system $I$ whose profile are given in Table 3.1. For the at money call options, we can calculate the ‘fair’ option premium $\bar{p} = 6.570512821$ by solving the equation

$$\mathbb{E}[C_I(0, \bar{p})] = \mathbb{E}[W_I(0, \bar{p})],$$

and checking that the common expected return equals to $\mathbb{E}[I]$. Assuming that we invest a constant amount of $100 on each trade and using the option premium $\bar{p} = 6.570512821$ calculated above we summarize the outcomes of using the original investment system along with replacing it by buying and writing at money calls in Table 3.2. As expected, in the end, the three investment systems all give us the same profit of $48. However, the processes of arriving at this result are drastically different. The smoothest is the strategy of writing covered calls, the roughest is buying calls, and the original investment system is sandwiched in between. Arguably, the strategy
of writing covered calls is preferred which will be singled out with the Kelly criterion. The strategy of buying call will actually wipe out the investor’s $100 initial capital resulting in a margin call at the second trade and will terminate the investment process with a loss of $184.78 unless additional funds are committed.

4 Option pricing intervals for a general investment system

We now turn to an analysis of the general case. It turns out that pictures don’t lie. What we observed in the previous section proves to be true in general with only a few minor amendments.

Throughout this section, $I = \{g_n, n = 1, \ldots, N\}$ is an investment system with $\mathcal{E}[I] \geq 0$. Again, we use $C_I(a, p)$ and $W_I(a, p)$ to denote the corresponding option investment systems of replacing each trade in $I$ by buying a call option and by writing a covered call option $a$ in the money with a premium $p$, respectively. Let us recall that

$$C_I(a, p) = \frac{(I + a)^+ - p}{p}$$

and

$$W_I(a, p) = \frac{p - a - (I + a)^-}{1 - p}.$$ 

Now our task is to compare these three investment systems by their efficiency indices $\gamma(I)$, $\gamma(C_I(a, p))$ and $\gamma(W_I(a, p))$. Let us start with two extreme cases. When $a \geq M_I^-$, the investment system of writing covered calls $W_I(a, p)$ contains identical elements $\frac{p-a}{1-p} > 0$ and

$$C_I(a, p) = \frac{I + a - p}{p}$$

so that $\gamma(W_I(a, p)) = +\infty$ and $\gamma(C_I(a, p)) = \gamma(I + a - p) \leq \gamma(I)$. This is to say that $W_I(a, p)$ provides an arbitrage opportunity while $C_I(a, p)$ is inferior to $I$. Similarly, when $a \leq -M_I^+$, the investment system of buying calls $C_I(a, p)$ contains identical elements $-1$ and

$$W_I(a, p) = \frac{I + p}{p}$$
so that $\gamma(W_I(a,p)) = \gamma(I + p) \geq \gamma(I)$. In other words $C_I(a,p)$ always loses money while $W_I(a,p)$ is always superior to $I$. This shows that deeply in the money or out of the money calls are in general unfavorable to buyers and favorable to covered call writers.

In what follows we assume that $a \in (-M_I^+, M_I^-)$ is fixed. Since $\gamma(C_I(a,p))$ and $\gamma(W_I(a,p))$ are functions of $p$, the relative merit of these investment systems are determined by $p$, the percentage premium for the option. As shown in [Zhu, 2006], in general, explicit formulae for $\gamma(C_I(a,p))$ and $\gamma(W_I(a,p))$ cannot be derived. Thus, we will focus on the qualitative properties of these two functions to determine critical values of $p$.

We start by identifying the values of $p$ such that $C_I(a,p)$ or $W_I(a,p)$ becomes invalid.

**Theorem 4.1.** Let $I$ be an investment system and let $a \in (-M_I^+, M_I^-)$. Define

$$
\begin{align*}
    u_a &= E[(I + a)^+] \\
    l_a &= a + E[(I + a)^-].
\end{align*}
$$

Then $u_a$ is the unique option premium such that $\gamma(C_I(a,u_a)) = 0$ and $l_a$ is the unique option premium such that $\gamma(W_I(a,l_a)) = 0$. Moreover,

$$
    u_a - l_a = E[I].
$$

*Proof.* It is easy to calculate that

$$
    E[C_I(a,p)] = \frac{E[(I + a)^+] - p}{p},
$$

and

$$
    E[W_I(a,p)] = \frac{p - a - E[(I + a)^-]}{1 - p}.
$$

Clearly $u_a$ is the unique option premium such that $E[C_I(a,u_a)] = 0$ and $l_a$ is the unique option premium such that $E[W_I(a,l_a)] = 0$. The conclusion of the theorem now follows immediately from Theorem 2.2. The formula (4.2) follows from a direct computation. Q.E.D.

Next, we show that for any investment system with an appropriate premium for the option, the effectiveness of the related investment systems of buying calls and writing covered calls can be balanced.
Theorem 4.2. For any investment system $I$ and $a \in (-M_I^+, M_I^-)$, there exists a unique option premium $p_a^* \in (l_a, u_a)$ such that
\[ \gamma(C_I(a, p_a^*)) = \gamma(W_I(a, p_a^*)). \]

Proof. As the maximum of differentiable functions both $\gamma(C_I(a, p))$ and $\gamma(W_I(a, p))$ are continuous functions of $p$. Since $\gamma(C_I(a, l_a)) > \gamma(W_I(a, l_a)) = 0$ and $0 = \gamma(C_I(a, u_a)) < \gamma(W_I(a, u_a))$ there exists $p_a^* \in (l_a, u_a)$ such that
\[ \gamma(C_I(a, p_a^*)) = \gamma(W_I(a, p_a^*)). \]

When $p \in (l_a, u_a)$, $E[C_I(a, p)] > 0$ so that the investment system $C_I(a, p)$ makes money. Moreover, for buying calls the lower the premium the better (strictly). Thus, $\gamma(C_I(a, p))$ is a strictly decreasing function on $(l_a, u_a)$. Similarly, $\gamma(W_I(a, p))$ is a strictly increasing function on $(l_a, u_a)$. Thus, the $p_a^*$ identified above is unique. Q.E.D.

We have seen examples of investment systems in Section 3 that this common value $\gamma(C_I(a, p_a^*)) = \gamma(W_I(a, p_a^*))$ was smaller or equal to $\gamma(I)$. Now we show that this is true in general. We will establish this fact as a corollary of the following more general result which says that, for a given investment system $I$ and any $a < 1$, there does not exist an option premium in the interval $[l_a, u_a]$ such that the related option investment systems of both buying calls and writing covered calls become better than $I$, the original investment system. Intuitively, the option investment systems of buying calls and writing covered calls are the two sides of a zero-sum game relative to the underlying investment system. Naturally we cannot expect both of them to be better than the original system. In fact, Example 3.1 in Section 3 shows that they can be both inferior to the original investment system, i.e., useful investment information could be lost in the process of trading options.

Theorem 4.3. For any investment system $I$, $a \in (-M_I^+, M_I^-)$ and $p \in [l_a, u_a]$,
\[ \min(\gamma(C_I(a, p)), \gamma(W_I(a, p))) \leq \gamma(I). \]

Proof. Let $I = \{g_n, n = 1, \ldots, N\}$. Let $\bar{s}_C$ and $\bar{s}_W$ be the best investment sizes of investment systems $C_I(a, p)$ and $W_I(a, p)$, respectively. Then
\[ \gamma(C_I(a, p)) = \frac{1}{N} \sum_{n=1}^{N} \ln \left( 1 + \frac{\bar{s}_C (g_n + a)^+ - p}{p} \right) \]
and 
\[ \gamma(W_I(a, p)) = \frac{1}{N} \sum_{n=1}^{N} \ln \left( 1 + \bar{s}_W \frac{p - a - (g_n + a)^-}{1 - p} \right) . \]

Set \( \alpha = p/\bar{s}_C \) and \( \beta = (1 - p)/\bar{s}_W \). Since the nature log is a concave function we have

\[ \frac{\alpha}{\alpha + \beta} \gamma(C_I(a, p)) + \frac{\beta}{\alpha + \beta} \gamma(W_I(a, p)) \]
\[ = \frac{1}{N} \sum_{n=1}^{N} \left( \frac{\alpha}{\alpha + \beta} \ln(1 + \bar{s}_C \frac{(g_n + a)^+ - p}{p}) + \frac{\beta}{\alpha + \beta} \ln(1 + \bar{s}_W \frac{p - a - (g_n + a)^-}{1 - p}) \right) \]
\[ \leq \frac{1}{N} \sum_{n=1}^{N} \ln \left( 1 + \frac{1}{\alpha + \beta} (g_n + a)^+ - p \right) + \frac{1}{\alpha + \beta} (p - a - (g_n + a)^-) \]
\[ = \frac{1}{N} \sum_{n=1}^{N} \ln \left( 1 + \frac{1}{\alpha + \beta} g_n \right) \leq \gamma(I). \]

Q.E.D.

**Corollary 4.4.** Let \( I \) be an investment system and let \( p^*_a \in (l_a, u_a) \) be the option premium stated in Theorem 4.2. Then

\[ \gamma(C_I(a, p^*_a)) = \gamma(W_I(a, p^*_a)) \leq \gamma(I). \]

Note that when \( p \leq l_a, W_I(a, p) \) loses money and when \( p \geq u_a, C_I(a, p) \) loses money. Thus, the only way that the option investment system \( C_I(a, p) \) (\( W_I(a, p) \)) can be better than \( I \) is at the expense of \( W_I(a, p) \) (\( C_I(a, p) \)) through a favorable option premium \( p \). Does such an option premium always exist? The answer is affirmative. For \( a \) outside the interval \((-M^+_I, M^-_I)\) we already observed this from our discussion on the extreme cases. The next theorem is about the nontrivial case when \( a \in (-M^+_I, M^-_I) \).

**Theorem 4.5.** Let \( I \) be an investment system and let \( a \in (-M^+_I, M^-_I) \). Then there exists a unique option premium \( p^*_a \) and a unique option premium \( p^*_a^- \) such that

\[ \gamma(C_I(a, p^*_a^-)) = \gamma(W_I(a, p^*_a^-)) = \gamma(I). \]
Consequently, when \( p < p_a^C \), \( \gamma(C_I(a, p)) > \gamma(I) \) and when \( p > p_a^W \), \( \gamma(W_I(a, p)) > \gamma(I) \).

Proof. When \( p \) approaches 0, for any trade in \( I \) with \( g_n + a > 0 \), the percentage gain of the corresponding trade in \( C_I(a, p) \), \( (g_n + a)/p - 1 \) approaches +\( \infty \) and for any trade in \( I \) with \( g_n + a \leq 0 \) the percentage loss of the corresponding trade in \( C_I(a, p) \) is always -1. Thus, \( \gamma(C_I(a, p)) \) approaches +\( \infty \) as \( p \) approaches 0. Since \( \gamma(C_I(a, p)) \) is a strictly decreasing function and \( \gamma(C_I(a, p^*_a)) \leq \gamma(I) \), there exists a unique \( p_a^C \in (0, p_a^*) \) such that \( \gamma(C_I(a, p_a^C)) = \gamma(I) \).

Similarly, when \( p = M_I^− \), every entry in

\[
W_I(a, p) = \{(p - a - (g_n + a)^+)/ (1 - p) : n = 1, \ldots, N\}
\]

is nonnegative. Thus, \( \gamma(W_I(a, p)) \) approaches +\( \infty \) as \( p \) approaches \( M_I^− \). Since \( \gamma(W_I(a, p)) \) is a strictly increasing function and \( \gamma(W_I(p_a^*)) \leq \gamma(I) \), there exists a unique \( p_a^W \in [p_a^*, M_I^−) \) such that \( \gamma(W_I(a, p_a^W)) = \gamma(I) \). Q.E.D.

While \( p_a^C \) and \( p_a^W \) are critical thresholds for \( C_I(a, p) \) and \( W_I(a, p) \) to out-perform the original investment system \( I \), we cannot expect explicit formulae for them. In fact, \( \gamma(I) \) itself does not even have an explicit formula when the number of trades \( N \) is large (see [Zhu, 2006]). Thus, they can only be calculated numerically and finding estimates for \( p_a^C \) and \( p_a^W \) is useful to narrow down a range for the numerical search.

Since \( \gamma(W_I(a, l_a)) = 0 \), trading option at premium \( l_a \) the writer of the covered call will lose money to the buyer of the call. Thus, we have reason to expect \( l_a \) to be a lower bound for \( p_a^C \). However, we need to make a detour to prove this fact. Let \( I = \{g_n, n = 1, \ldots, N\} \) be an investment system. Let \( C_I(a, p) \) be the corresponding investment system of buying calls at premium \( p \). Then \( C_I(a, p) = \{(g_n + a)^+/ p - 1, n = 1, \ldots, N\} \). Now let us consider the investment system \( pC_I(a, p) \). It is easy to see that \( pC_I(p) = \{(g_n + a)^+ - p, n = 1, \ldots, N\} \). The investment system \( pC_I(a, p) \) has the following meaning: for each trade in \( I \), dividing the capital (which is 1 in our standardization) into two parts of \( 1 - p \) and \( p \) the investment system uses \( p \) to purchase a corresponding call option \( a \) in the money with a premium \( p \) and reserves the remaining \( 1 - p \) in cash. We will call \( pC_I(a, p) \) the investment system of buying option with full capital reserve corresponding to \( I \) and use it as a bridge to reach our anticipated conclusion that \( l_a \) is a lower bound for \( p_a^C \).
**Lemma 4.6.** Let $I$ be an investment system, let $a \in (-M^+, M^-)$ and let $pC_I(a, p)$ be the investment system of buying option with full capital reserve corresponding to $I$. Let $p$ be an option premium such that 

$$\mathcal{E}[I] = \mathcal{E}[pC_I(a, p)].$$

Then, for any $s \geq 0$,

$$f_I(s) \leq f_{pC_I(a, p)}(s).$$

**Proof.** We prove by induction on the number of trades $N$. For $N = 2$, the conclusion follows directly from Lemma 2.5. Assume that the theorem holds for investment systems with $N - 1$ trades. We proceed to prove it for investment system $I$ with $N$ trades.

Let $I = \{g_n, n = 1, \ldots, N\}$. Assume without loss of generality that

$$g_1 \geq g_2 \geq \ldots \geq g_k \geq -a > g_{k+1} \geq \ldots \geq g_N.$$ 

Then $pC_I(a, p) = \{c_n, n = 1, \ldots, N\}$ where 

$$c_n = \begin{cases} g_n + a - p & \text{if } n = 1, \ldots, k \\ -p & \text{if } n = k + 1, \ldots, N. \end{cases}$$

The condition $\mathcal{E}[I] = \mathcal{E}[pC_I(a, p)]$ implies that $p = a + \mathcal{E}[(I + a)^-] \geq a$ and that

$$0 = \sum_{n=1}^{N} (g_n - c_n)$$

$$= \sum_{n=1}^{k} (p - a) + \sum_{n=k+1}^{N} (g_n + p)$$

$$\geq \sum_{n=k+1}^{N} (g_n + p).$$

It follows that $g_N + p \leq 0$. Now we construct a new investment system with $N - 1$ trades:

$$I' = \{g_2, \ldots, g_{N-1}, g_N + p - a\}.$$

Since $(g_N + p - a) + a = g_N + p \leq 0$ the investment system of buying call option with full capital reserve corresponding to $I'$ in the money by $a$ with a premium $p$ is $pC_{I'}(a, p) = \{c_2, \ldots, c_{N-1}, c_N\}$. 

23
Since \( g_1 + a - p = c_1 \), we can check that
\[
\mathcal{E}[I'] - \mathcal{E}[pC_I(a, p)] = \mathcal{E}[I'] - \mathcal{E}[pC_I(a, p)] + \frac{(g_1 + a - p) - c_1}{N - 1} = \frac{N}{N - 1} (\mathcal{E}[I] - \mathcal{E}[pC_I(a, p)]) = 0.
\]
By the induction hypothesis we have, for \( s \geq 0 \),
\[
f_I(s) \leq f_{pC_I(a, p)}(s). \quad (4.3)
\]
Since \( g_1 + a - p = c_1 \geq 0 \geq g_N + p - a \), we have \( g_1 - g_N - 2(p - a) \geq 0 \).
Using inequality (4.3) and Lemma 2.5 we have
\[
f_I(s) = \frac{1}{N} \left( \ln(1 + sg_1) + \ln(1 + sg_2) + \ldots + \ln(1 + sg_{N-1}) + \ln(1 + sg_N) \right)
\leq \frac{1}{N} \left( \ln(1 + s(g_1 + a - p)) + \ln(1 + sg_2) + \ldots + \ln(1 + sg_{N-1}) + \ln(1 + s(g_N + p - a)) \right)
= \frac{N - 1}{N} \left( f_{pC_I(a, p)}(s) + \frac{1}{N - 1} \ln(1 + sc_1) \right)
= \frac{1}{N} \sum_{n=1}^{N} \ln(1 + sc_n) = f_{pC_I(a, p)}(s),
\]
which completes the induction proof. Q.E.D.

**Theorem 4.7.** Let \( I \) be an investment system and \( a \in (-M_I^+, M_I^-) \). Then
\[
p_a^C \geq l_a.
\]

**Proof.** Solving equation
\[
\mathcal{E}[pC_I(a, p)] = \mathcal{E}[I]
\]
we have \( p = l_a \). It follows from Lemma 4.6 that, for any \( s \geq 0 \),
\[
f_I(s) \leq f_{pC_I(a, p)}(s).
\]
Thus, $\gamma(I) \leq \gamma(l_a C_I(a, l_a)) = \gamma(C_I(a, l_a))$ by (2.5). Since $\gamma(C_I(a, p_a^w)) \leq \gamma(I)$ we have $p_a^C \in [l_a, p_a^w]$. This establishes $l_a$ as a lower bound for $p_a^C$. Q.E.D.

Similarly we can expect $u_a$ to be an upper bound for $p_a^W$. This is true but we can do better in showing that $\bar{p}_a := u_a/(1 + \mathcal{E}[I])$ is an upper bound for $p_a^W$. First, we record that $\bar{p}_a$ is the value at which the expected returns of all the three investment systems $I, C_I(a, \bar{p}_a)$ and $W_I(a, \bar{p}_a)$ coincide.

**Lemma 4.8.** Let $I$ be an investment system and let $a < 1$. Define $\bar{p}_a := u_a/(1 + \mathcal{E}[I])$. Then

$$\mathcal{E}[I] = \mathcal{E}[C_I(a, \bar{p}_a)] = \mathcal{E}[W_I(a, \bar{p}_a)]$$

**Proof.** This follows from a simple direct computation. Q.E.D.

**Theorem 4.9.** Let $I$ be an investment system, let $a \in (-M_i^+, M_i^-)$ and let $\bar{p}_a := u_a/(1 + \mathcal{E}[I]) = \mathcal{E}[(I + a)^+]/(1 + \mathcal{E}[I])$. Then

$$p_a^W \leq \bar{p}_a.$$  

**Proof.** We need to show that $\gamma(I) \leq \gamma(W_I(a, \bar{p}_a))$. This follows from the stronger result in our next theorem. Q.E.D.

We now show that for the option premium $\bar{p}_a$ that balances the expected returns of the investment systems $I$ and $W_I(a, \bar{p}_a)$, in terms of investment performances, at any nonnegative investment size, $W_I(a, \bar{p}_a)$ is the best followed by $I$ and $C_I(a, \bar{p}_a)$ is always the worst.

**Theorem 4.10.** Let $I$ be an investment system, let $a \in (-M_i^+, M_i^-)$ and let $\bar{p}_a$ be the option premium such that $\mathcal{E}[I] = \mathcal{E}[W_I(a, \bar{p}_a)]$. Then, for any $s \geq 0$,

$$f_{C_I(a, \bar{p}_a)}(s) \leq f_I(s) \leq f_{W_I(a, \bar{p}_a)}(s).$$

Consequently,

$$\gamma(C_I(a, \bar{p}_a)) \leq \gamma(I) \leq \gamma(W_I(a, \bar{p}_a)).$$

**Proof.** We observe that due to the concavity of the nature log function, for any $p \in [a^+, 1]$,

$$pf_{C_I(a, p)}(s) + (1 - p)f_{W_I(a, p)}(s) \leq f_I(s).$$

Thus, we need only to show that

$$f_I(s) \leq f_{W_I(a, \bar{p}_a)}(s).$$

25
We prove by induction on the number of trades $N$. For $N = 2$, the conclusion follows directly from Lemma 2.5. Assume that the theorem holds for investment systems with $N - 1$ number of trades. We proceed to prove it for investment system $I$ with $N$ trades.

Let $I = \{g_n, n = 1, \ldots, N\}$. Assume without loss of generality that

$$g_1 \geq g_2 \geq \ldots \geq g_j \geq \frac{\bar{p} - a}{1 - \bar{p}} > g_{j+1} \geq \ldots \geq g_k \geq -a > g_{k+1} \geq \ldots \geq g_N.$$ 

Then $W_I(a, \bar{p}) = \{w_n, n = 1, \ldots, N\}$, where

$$w_n = \begin{cases} \frac{\bar{p} - a}{1 - \bar{p}} & n = 1, \ldots, k \\ \frac{\bar{p} + g_n}{1 - \bar{p}} & n = k + 1, \ldots, N. \end{cases}$$

For $n = k + 1, \ldots, N$, since $g_n \in [-1, 0)$ we have $\frac{\bar{p}(1 + g_n)}{1 - \bar{p}} \in [0, \bar{p}/(1 - \bar{p})]$. Let $h = w_N - g_N = \frac{\bar{p}(1 + g_N)}{1 - \bar{p}}$. The condition $\mathcal{E}[I] = \mathcal{E}[W_I(a, \bar{p}_a)]$ implies that

$$0 = \sum_{n=1}^{N} (w_n - g_n)$$

$$= \sum_{n=1}^{j} \left( \frac{\bar{p} - a}{1 - \bar{p}} - g_n \right) + \sum_{n=j+1}^{k} \left( \frac{\bar{p} - a}{1 - \bar{p}} - g_n \right) + \sum_{n=k+1}^{N} \frac{\bar{p}(1 + g_n)}{1 - \bar{p}}$$

$$\geq \sum_{n=1}^{j} \left( \frac{\bar{p} - a}{1 - \bar{p}} - g_n \right) + h,$$

We can write $h = h_1 + h_2 + \ldots + h_j$ with $h_n \geq 0$ satisfying $\frac{\bar{p} - a}{1 - \bar{p}} - g_n + h_n \leq 0$, for $n = 1, \ldots, j$. Using the $h_n$’s we construct a new investment system with $N - 1$ trades:

$$I' = \{g_1 - h_1, g_2 - h_2, \ldots, g_j - h_j, g_{j+1}, \ldots, g_{N-1}\}.$$ 

Since $g_n + a - h_n \geq (\bar{p}_a - a)/(1 - \bar{p}_a) + a = \bar{p}_a(1 - a)/(1 - \bar{p}_a) > 0$ for $n = 1, \ldots, j$ the corresponding investment system of writing covered calls $a$ in the money with premium $\bar{p}_a$ is

$$W_{I'}(a, \bar{p}_a) = \{w_1, w_2, \ldots, w_j, \ldots, w_{N-1}\}.$$
Since $w_N = g_N + h$, we can check that
\[
\mathcal{E}[I'] - \mathcal{E}[W_{I'}(a, \bar{p}_a)] \\
= \mathcal{E}[I'] - \mathcal{E}[W_{I'}(a, \bar{p}_a)] + \frac{g_N + h - w_N}{N - 1} \\
= \frac{N}{N - 1} (\mathcal{E}[I] - \mathcal{E}[W_I(a, \bar{p}_a)]) = 0.
\]

By the induction hypothesis we have, for $s \geq 0$,
\[
f_{I'}(s) \leq f_{W_{I'}(a, \bar{p}_a)}(s). \tag{4.4}
\]

Note that, for $n = 1, 2, \ldots, j$, we have
\[
g_n - h_n \geq \frac{\bar{p} - a}{1 - \bar{p}} \geq w_N = g_N + h \geq (g_N + h_1 + \ldots + h_{n-1}) + h_n.
\]

Using Lemma 2.5 $j$ times we have
\[
f_I(s) = \frac{1}{N} \left( \ln(1 + s g_1) + \ln(1 + s g_2) + \ldots + \ln(1 + s g_j) \\
+ \ln(1 + s g_{j+1}) + \ldots + \ln(1 + s g_{N-1}) + \ln(1 + s g_N) \right) \\
\leq \frac{1}{N} \left( \ln(1 + s (g_1 - h_1)) + \ln(1 + s g_2) + \ldots + \ln(1 + s g_j) \\
+ \ln(1 + s g_{j+1}) + \ldots + \ln(1 + s g_{N-1}) + \ln(1 + s (g_N + h_1)) \right) \\
\leq \frac{1}{N} \left( \ln(1 + s (g_1 - h_1)) + \ln(1 + s (g_2 - h_2)) + \ldots + \ln(1 + s g_j) \\
+ \ln(1 + s g_{j+1}) + \ldots + \ln(1 + s g_{N-1}) + \ln(1 + s (g_N + h_1 + h_2)) \right) \\
\ldots \ldots \\
\leq \frac{1}{N} \left( \ln(1 + s (g_1 - h_1)) + \ln(1 + s (g_2 - h_2)) + \ldots + \ln(1 + s (g_j - h_j)) \\
+ \ln(1 + s g_{j+1}) + \ldots + \ln(1 + s g_{N-1}) + \ln(1 + s (g_N + h)) \right) \\
= \frac{N - 1}{N} \left( f_{I'}(s) + \frac{1}{N - 1} \ln(1 + s (g_N + h)) \right).
\]

Combining the above with the inequality (4.4) and using $g_N + h = w_N$ we
have
\[
I(s) \leq \left( f_I(s) + \frac{1}{N-1} \ln(1 + s(g_N + h)) \right)
\leq \left( f_{I'}(s) + \frac{1}{N-1} \ln(1 + sw_N) \right)
= \frac{1}{N} \sum_{n=1}^{N} \ln(1 + sw_n) = f_{I'(a, \bar{p}_a)}(s),
\]
which completes the induction proof. Q.E.D.

To summarize we have

\textbf{Theorem 4.11.} Let $I$ be an investment system with a positive expected return and let $a \in (-M^+_I, M^-_I)$. Then there exists three critical option premiums $p_a^C$, $p_a^W$ and $p_a^*$ satisfying
\[
l_a \leq p_a^C \leq p_a^* \leq p_a^W \leq \bar{p}_a \leq u_a,
\]
where $l_a = a + \mathcal{E}[(I + a)^-]$ and $u_a = \mathcal{E}[(I + a)^+]$. They have the following properties:

(a) when $p \in [p_a^C, p_a^W]$, 
\[
\max(\gamma(W_I(a, p)), \gamma(C_I(a, p))) \leq \gamma(I).
\]
In particular,
\[
\gamma(C_I(a, p^*)) = \gamma(W_I(a, p^*)) \leq \gamma(I).
\]
(b) when $p \in [l_a, p_C]$, 
\[
\gamma(C_I(a, p)) \geq \gamma(I) \geq \gamma(W_I(a, p)).
\]
(c) when $p \in [p_W, u_a]$, 
\[
\gamma(W_I(a, p)) \geq \gamma(I) \geq \gamma(C_I(a, p)).
\]

Properties of these investment systems outside the interval $[l_a, u_a]$ are less important. We record them below for completeness. The justifications for these facts have already appeared in the discussions.
Theorem 4.12. Let $I$ be an investment system with a positive expected return and let $a < 1$. Then

(a) when $p \geq u_a$, the investment system $W_I(a, p)$ improves $I$ while the investment system $C_I(a, p)$ loses money, and

(b) when $p \leq l_a$, the investment system $C_I(a, p)$ improves $I$ while the investment system $W_I(a, p)$ loses money.

The efficiency indices of these investment systems has the following limiting properties:

\[
\lim_{p \to 0} \gamma(C_I(a, p)) = +\infty, \quad (4.6) \\
\lim_{p \to 0} \gamma(W_I(a, p)) = +\infty, \quad (4.7) \\
\lim_{p \to M_I^+} \gamma(C_I(a, p)) = +\infty, \quad (4.8) \\
\lim_{p \to M_I^-} \gamma(W_I(a, p)) = +\infty. \quad (4.9)
\]

Moreover, when $p \in (0, l_a]$, \(\gamma(C_I(p)) \geq \gamma(W_I(p))\).

Thus, the picture we observed in Figure 2 is quite accurate in revealing the general pattern. The only possible alternative scenario is when $M_I^+ \geq M_I^-$ the graphs of $\gamma(C_I(a, p))$ and $\gamma(W_I(a, p))$ will have no intersection for $p > u_a$.

5 Compatibility with pricing by a replicating portfolio

When the payoff of the option can be exactly replicated by a portfolio of stocks and bonds, assuming no arbitrage one can derive the valuation of the option by either directly calculating the cost of the replicating portfolio following the idea of [Black and Scholes, 1973, Merton, 1973] or use the risk neutrality argument in [Cox and Ross, 1976]. We show that the option pricing interval $[p^n_C, p^n_W]$ is compatible to both methods.

Given an investment system $I = \{g_n, n = 1, \ldots, N\}$, the possible values of the stock at time 1 are $\{1 + g_n, n = 1, \ldots, N\}$ and the possible payoffs of a call option $a$ in the money are $\{(g_n + a)^+, n = 1, \ldots, N\}$. Suppose that the
payoff of the call option can be replicated by a portfolio of $\alpha$ stocks and $\beta$ bonds in our one period economy. Then, for $n = 1, \ldots, N$,

$$\alpha(1 + g_n) + \beta = (g_n + a)^+,$$

and the pricing of the option by this replicating portfolio is $p_r = \alpha + \beta$. We can easily verify that

$$C_I(a, p_r) = \frac{(I + a)^+ - p_r}{p_r} = \frac{\alpha(1 + I) + \beta - p_r}{p_r} = \frac{\alpha}{p_r}I$$

and

$$W_I(a, p_r) = \frac{p_r - a - (I + a)^-}{1 - p_r} = \frac{p_r - a + (I + a) - (I + a)^+}{1 - p_r} = \frac{1 - \alpha}{1 - p_r}I.$$

By (2.5) we have

$$\gamma(C_I(a, p_r)) = \gamma(W_I(a, p_r)) = \gamma(I)$$

and, therefore, $p_r = p^C_a = p^*_a = p^W_a$. In other words the thresholds $p^C_a$ and $p^W_a$ both coincide with $p_r$, the option valuation derived by calculating the cost of the replicating portfolio.

We now turn to the risk neutrality argument in [Cox and Ross, 1976]. The idea is that if we can exactly replicate the payoff of the option with a portfolio involving stocks and bonds then the option can be hedged without any risk. Thus, its valuation is the same for all investors regardless of their individual risk tolerance. In particular, we can evaluate the option in an idealized risk neutral economy where investors do not ask for a risk premium for risky investments and, therefore, all investments have the same expected rate of return that equals to that of the risk-less bond. Now in our setting of using the real value, the rate of return of the risk-less bond is 0. Applying our result in this risk neutral economy, we assume that the value of the stock at time 1 satisfies the risk neutral probability distribution so that $\mathcal{E}[I] = 0$. It follows that $u_a = l_a$ and, therefore, all the critical option premiums in (4.5) coincide. In particular, $p^W_a = p^C_a = u_a = \mathcal{E}[(I + a)^+]$. Note that $\mathcal{E}[(I + a)^+]$ is exactly the option valuation in the risk neutral economy.
6 Concluding remarks

We studied investment system specific option valuations in a one period economy. Using the Kelly criterion we find that for each investment system \( I \) with a nonnegative expected return and a call option \( a \) in the money, there are two thresholds of call option premiums, \( p^C_a \leq p^W_a \), such that buying (writing covered) calls will improve the original investment system when the option premium \( p < p^C_a (p > p^W_a) \). Moreover, bounds for \( p^C_a \) and \( p^W_a \) are provided. Furthermore the investment systems of buying and writing calls cannot be simultaneously better than the original investment system and, in fact, they are both inferior to the original investment system \( I \) when the option premium belongs to the interval \([p^C_a, p^W_a]\). Finally, the option premium that equalize the expected return of the investment systems of buying and writing calls related to the original investment system favors the strategy of writing covered calls.

We derived these results in terms of real prices and percentage gains adjusted for the rate of return on risk-free assets. This is for mathematical rigor. In practice, the barrier for moving capital between an investment account and a fixed income investment instrument that approximates the risk-free assets is high while interest for unused capital in an investment account is negligible. Thus, when using the results here in practice it is more appropriate to calculate in terms of nominal prices and percentage gains.

We compare investment systems using the Kelly criterion. This has the advantage of yielding option pricing intervals that are consistent with option valuations derived by using replicating portfolio or a risk neutrality argument when they are applicable. This is perhaps due to the information theoretical nature of the Kelly criterion, and it highlights the relevance of information theoretical methods in the analysis of financial problems.

The analysis here is restricted to a one period economy which provides guide to simple implementable option investment strategies. Similar analysis for multiple period economy and, in the limit, continuous economy is an interesting direction for further research which may provide useful insight in both theory and practice.
References


