Vector majorization and a robust option replacement trading strategy *

Q. J. Zhu
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA
e-mail: zhu@wmich.edu

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Abstract

We show that vector majorization and its related preference sets can be used to establish useful option pricing bounds for a robust option replacement investment strategy. This robust trading strategy can help to overcome some of the difficulties in implementing arbitrage option trading strategies when there exists model inaccuracy.

Key Words. Robust option trading strategy, stability, majorization, and convex optimization.

* *Dedicated to Boris Mordukhovich on his 60th birthday*

1 Introduction

Option valuation draws increasing attention due to the prosperity of financial option products and other forms of contingent claims in the financial industry that can be represented using options. One of the most influential paradigm in this area is the method of valuing options by self-financing dynamic hedging proposed in the pioneering work of Black and Scholes [4] and Merton [20]. The idea is to replicate the payoff of the option using a self-financing portfolio of the underlying investment instrument and a bond generating the market risk-free rate. Under the no arbitrage principle, the cost of the portfolio will be the fair price of the option. However, we show in the next section by a simple example that the portfolio replicating strategy is unstable with respect to model inaccuracy, which is inevitable in practice (see e.g. [11]). This instability seems to be the root of the missteps of many hedge funds [22, 15].

Motivated by the example in Section 2, we study a robust trading strategy. Suppose an investor has a profitable investment system for optionable investment instruments in a one period economy. Our strategy allows only to either invest in the original instrument, buy call options or write covered call options. The decision on which one to use is determined by whether the price of the option is 'right' so that using the options to replace the original investment instrument will improve the investment system. To compare the relative merit of different investment systems, we use the maximum utility of an investment system under the best money management with respect to a general utility function. This method naturally leads to a pricing mechanism for the options. Part of the results has been reported in [28].

To explain our results, let us focus on an investment system for optionable stocks. We assume that the investment system contains one (representative) stock whose price at the beginning of the period is standardized to 1 and the price of the stock at the end of the period is a random variable X in terms of the real value discounted for the market risk-free interest rate. Then $x = X - 1$ is the net percentage return of the stock. The investor knows the subjective probability distribution of x (more precisely, its approximation), and the investment system is profitable so that $\mathcal{E}[x] \geq 0$. In this case, only call options could possibly improve the investment system. For simplicity, we illustrate by using the at money call whose strike price is also 1. To trade a call option, there are two sides: the buyer of the call and the writer of the covered call. We can view these two sides as engaged in a zero sum game relative to the underlying stock. Thus, heuristically, if one side is doing better than the stock, then that is probably at the expense of the other side due to a favorable option price. This proves to be true. In addition we find that, in general, the option trading is inefficient in the sense that there exists a pricing interval inside which both the buyer and the writer of the option underperform the stock. More precisely, there exist two critical call option premiums: $p^C \leq p^W$ having the following properties. When the option premium $p < p^C$, buying call options will improve the investment system and when $p > p^W$, writing covered call options will improve the investment system. For any option premium p that belongs to the option pricing interval $[p^C, p^W]$ the original investment system gives the best result. The fact that, in general, $p^C < p^W$ indicates that buying calls and writing calls should be treated as different alternatives and should be priced differently.

The pricing thresholds p^C and p^W depend on the utility functions and can generally only be numerically estimated. It turns out that two easily determined critical price levels provide bounds for these thresholds as below

$$l \leq p^C \leq p^W \leq \bar{p}. \tag{1.1}$$

Here, l and $\bar{p}$ are defined by $l = \mathcal{E}[(x)^-]$ and $\bar{p} = \mathcal{E}[(x)^+]/\mathcal{E}[1+x] \leq \mathcal{E}[(x)^+]$, respectively, with $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. We note that $\bar{p}$ is the option premium that balances the expected percentage returns of buying calls and writing covered calls. This common expected percentage return also coincides with $\mathcal{E}[x]$, the expected percentage return of the original investment system. The general pattern in (1.1) also holds for in the money and out of the money call options. The intuition that helps us to arrive at those bounds is that given an investment system, at the option price that equalizes the expected returns of the stock investment system and its related option replacement systems, the investment system of buying call is always more volatile than the underlying investment system and the investment system of writing covered call is always smoother. Mathematically, we establish that under this equalizing price, viewing as vectors, the investment system of buying calls majorizes (see Section 5 and [1, 3]) the underlying investment system which in turn majorizes the investment system of writing calls. The lower bound l is also established with the help of majorization. Since majorization is a characterization of second degree stochastic dominance for random variables in discrete sample spaces, l and $\bar{p}$ (1.1) can be viewed as lower and upper bounds for option pricing in the sense of stochastic dominance. Such bounds have been studied for the pricing of call options in Levy

[18] (writing call options was not considered as an independent investment alternative there) refining the earlier work of Merton [20]. Using a continuous model, under the additional assumption that the probability distribution functions for both the stock and the option are invertible, Levy [18] established the estimate $p^C \leq \bar{p}$ and derived a lower bound for p^C different from l. Whether the stochastic dominance bounds in [18] and here are special cases of a more general pattern seems to be an interesting problem.

In a complete market, the two main methods of pricing options are the replicating portfolio pioneered in [4, 20] and the risk neutrality argument proposed in [9]. We show that our method is compatible to both in the sense that if the option payoff can be replicated, then $p^C = p^W$ and the common value coincides with the option valuation derived by the aforementioned classical methods. However, we emphasize that, even in this case, the implication for implementation is different: our method calls for a simple option replacement strategy which is robust with respect to model inaccuracies.

The trading strategy and the related pricing mechanism discussed in this paper also apply to incomplete markets and they produce an option pricing interval $[p^C, p^W]$. Option pricing intervals in incomplete markets have been studied using other methods. For example, Ledoit [17] and Bernardo and Ledoit [2] relied on the Sharpe ratio as the comparing criterion; and Cochrane and Saá Requejo [7] used discount factors motivated by the Sharpe ratio. Those pricing intervals are not necessarily compatible to option valuations derived by the replicating portfolio or the risk neutral argument when they are applicable.

There is an extensive literature studying portfolio optimization involving options. One can determine whether a price of the option is favorable by whether at that price the option is included in the optimal portfolio. Recent studies by Carr and Madan [6] and by Iihan, Johsson and Sircar [14] are examples of research in this direction and both discuss one period models. Also relevant is the utility indifference pricing discussed in [13, 21, 23, 25]. This method determines the price of an option as one that does not change the maximum utility when it is added to an optimal portfolio of stocks and bonds where the optimization is over the mix of stocks and bonds only. These methods usually lead to trading strategies that are 'optimal' in a certain sense but are not necessarily robust with respect to model inaccuracy.

Recently, robust utility maximization of portfolios involving options were studied in [8, 24, 12] by Cont, Schied, and Hernandez-Hernandez and Schied. They account for the model uncertainly with different probability measures of the underlying sample space representing the economic status. Somewhat similar is an approach taken by Carr, Geman and Madan in [5]. There, a set of probability measures were used to represent the beliefs of multiple market participants, and an acceptable price for an option is determined by its acceptability to all the players. Besides different representations of the model inaccuracies, our analysis here is also different from those in [8, 24, 12, 5] in that rather than pursuing the optimal strategy under a worst case scenario, we focus on a simple robust strategy that is easy to implement.

The remainder of the paper is arranged as follows. In the next section, we analyze a simple motivating example; in Section 3, we layout the finance model; Section 4 is devoted to derive option pricing intervals; Section 5 establishes estimates for the pricing interval using vector majorization; Section 6 discusses the option replacement strategy and its robustness; a concrete illustrative example is given in Section 7; Section 8 addresses the relationship between our strategy and methods for pricing options by replicating portfolio and by using a risk neutrality argument; concluding remarks are contained in Section 9.

2 A motivating example

We consider a common stock and its related call options. Standardize the current price of the stock to be 1 and use x to represent the percentage gain of the stock at the maturity of the option and consider the at money call option whose payoff is related to the stock price and is given by $x^+ := \max(x, 0)$. The option pricing problem asks what is a fair price of the call option. To illustrate the idea of no arbitrage pricing and its limitations, we consider a simple situation where no trading before the maturity of the option is allowed. We further assume that, at the maturity of the option, the possible prices and payoff of the stock and the option are given in the following diagram (with the probability of each pathway indicated):

$$
\begin{tikzcd}
\text{Stock} & \text{Maturity} & \text{Option} & \text{Maturity} \\
 & 1+a & & a \\
1 \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & & c \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\
 & 1-b & & 0
\end{tikzcd}
$$

The no arbitrage pricing method argues as below. Constructing a portfolio consists of $a/(a+b)$ shares of stock and $a(b-1)/(a+b)$ cash. It is easy to check that the payoff of this portfolio exactly matches that of the option in both scenarios and, therefore, the fair price of the option must be

$$c = \frac{a}{a+b} + \frac{a(b-1)}{a+b} = \frac{ab}{a+b}.$$

Now suppose the actual option price is $c + dc$. If $dc > 0$ we can buy the portfolio and sell the option to earn dc without any risk. In this case dc is called an arbitrage profit. If $dc < 0$ then we can do the opposite to gain an arbitrage profit. Thus, in theory one should always take the arbitrage strategy when the price of the option deviates from its theoretical value c. However, in practice, no one can precisely predict the percentage gain x at the time of buying the stock. Suppose the actual price of the option and the payoff of the stock and the option at the maturity are as below:

$$\begin{tikzcd} \text{Stock} & \text{Maturity} \\ & 1+a+da \\ 1 \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 1-b-db \end{tikzcd} \qquad \begin{tikzcd} \text{Option} & \text{Maturity} \\ & a+da \\ c+dc \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 0 \end{tikzcd}$$

Let us assume $dc > 0$ and discuss what happens to a arbitrage strategy according to the anticipated stock payoff of $1+a$ and $1-b$ with probabilities of π and $1-\pi$, respectively. In this situation we should purchase the replicating portfolio and sell the option. Note that to illustrate the replicating effect in the portfolio we borrowed cash in the amount of $a(1-b)/(a+b)$ which is of course not necessary when we actually conduct the trades. The actual transactions will be to purchase $a/(a+b)$ share of stock and to sell 1 call option with a net investment of $a(1-b)/(a+b) - dc$. The payoff and percentage gain of this investment is

$$\begin{tikzcd} \text{Cost} & \text{Payoff} & \text{Percetage gain} \\ & \frac{a(1-b)+ada}{a+b} & \frac{ada+(a+b)dc}{a(1-b)-(a+b)dc} \\ \frac{a(1-b)}{a+b} - dc \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & & \\ & \frac{a(1-b)-adb}{a+b} & \frac{-adb+(a+b)dc}{a(1-b)-(a+b)dc} \end{tikzcd}$$

As expected, when $da = db = 0$, we arrive at an arbitrage with percentage gain of $\frac{(a+b)dc}{a(1-b)-(a+b)dc}$ regardless of what happens. However, in reality, typically the magnitudes of da and db are larger than that of dc which could result in a situation of net loss in both cases. The example we presented here can be regarded as a one period case of the general binomial model of

Cox, Ross, and Rubinstein [10]. Taking limits as the number of periods approaches to infinity, the binomial model yields the celebrated Black-Scholes option pricing model [4]. Although the models in [10] are more elaborate, the same instability exists in each step. This instability is a critical difficulty in implementing the arbitrage trading strategy in practice.

To see the cause of this instability and method for reduction, let us assume the stock has a positive expected return $e = \pi a + (1-\pi)(-b) > 0$ and consider a general portfolio of α share of stocks and β shares of the call option with a cost of $\alpha + \beta(c + dc)$. Since the stock has a positive expected return we need only to consider $\alpha + \beta(c + dc) > 0$. The payoff and percentage gain of this portfolio is

Cost		Payoff	Percetage gain
$\alpha + \beta(c + dc)$	π	$a(\alpha + \beta\frac{a}{a+b}) + (\alpha + \beta)da - \beta dc$	$\frac{a(\alpha+\beta\frac{a}{a+b})+(\alpha+\beta)da-\beta dc}{\alpha+\beta(c+dc)}$
	$1 - \pi$	$-b(\alpha + \beta\frac{a}{a+b}) - \alpha db - \beta dc$	$\frac{-b(\alpha+\beta\frac{a}{a+b})-\alpha db-\beta dc}{\alpha+\beta(c+dc)}$

We can see that the strategy of using a replicating portfolio corresponds to $\alpha + \beta\frac{a}{a+b} = 0$. This completely removes the uncertainty related to the stock together with its potential of a positive return. Under this strategy, when the model has errors da and db, which is typically larger than dc in magnitude, they dominate the outcome. This leads to the instability that we observed above. Since model inaccuracy always exists, in practice, we should regard every investment strategy as risky.

Now let us assume $|da|$ and $|db|$ are bounded by $\delta > dc$, e is much larger than δ, and dc and try to find a portfolio that works best under the worst case scenario:

Cost		Payoff	Percetage gain
$\alpha + \beta(c + dc)$	π	$a(\alpha + \beta\frac{a}{a+b}) - \|\alpha + \beta\|\delta - \beta dc$	$\frac{a(\alpha+\beta\frac{a}{a+b})-\|\alpha+\beta\|\delta-\beta dc}{\alpha+\beta(c+dc)}$
	$1 - \pi$	$-b(\alpha + \beta\frac{a}{a+b}) - \|\alpha\|\delta - \beta dc$	$\frac{-b(\alpha+\beta\frac{a}{a+b})-\|\alpha\|\delta-\beta dc}{\alpha+\beta(c+dc)}$

Note that the percentage gains are homogeneous with respect to (α, β) and proportionally changing the percentage gain (loss) yields equivalent portfolios. To get the best expected return under the worst case scenario, we consider the optimization problem of maximizing

$$f(\alpha, \beta) := e(\alpha + \beta\frac{a}{a+b}) - (\pi|\alpha + \beta| + (1-\pi)|\alpha|)\delta - \beta dc,$$

under the constraint

$$|\alpha| + |\beta| = 1.$$

Since f is piecewise linear, the candidates for solutions are the corner points of the constraint set $|\alpha| + |\beta| = 1$ and those satisfy $\alpha + \beta = 0$ or $\alpha = 0$. A quick enumeration leads to three possibilities

$$f(1/2, -1/2) = \frac{eb}{a+b} + dc - (1-\pi)\delta,$$

$$f(0,1) = \frac{ea}{a+b} - dc - \pi\delta,$$

and

$$f(1,0) = e - \delta$$

corresponding to writing call options, buying call options and holding the stocks. Thus, we can expect that a robust trading strategy is to take one of these positions at a time. The question is what criterion should we use to trigger the switch. Clearly, comparing the expected return is not advisable because investment opportunities with the same expected return may involve different risk levels. Thus, we must use a criterion that accounts for the risk aversion of the investors, which we discuss in the next section.

3 The financial model

We use a trade as the basic unit of an investment system. A *trade* is the process of acquiring an investment instrument using the investment capital and subsequently liquidating it and returning the proceeds to the investment capital. We define an *investment system* as a set of trades. We will model this as a one period economy containing a risky investment instrument – the investment system and a risk-free asset. We re-scaled the duration of each trade in the investment system to $[0, 1]$, while using random variable x to represent the real return of a typical trade, relative to the risk-free asset, so that we will assume below that the risk-free asset in this economy is simply cash. This way, our investment system is characterized by x.

For a given investment system, we approximate the distribution of x by using historical trades of the system in a relevant period. Thus, if the historical data contains N trades, x takes N values (maybe repeated) corresponding to the net return of trades, with each of them having a probability of $1/N$

of occurring. Labeling the trades in the trading system by $\{1, 2, \ldots, N\}$, we can often conveniently regard x and, therefore, the corresponding investment system as a vector $x = (x_1, \ldots, x_N) \in R^N$. In what follows, we will use both views.

Next, we consider replacing trades in an investment system by corresponding options in this one period economy. This means that we assume the maturity of these options is also at time 1. We assume that the investment system is at least not losing money, that is, $\mathcal{E}[x] \geq 0$ so that we need only to consider call options. Again we standardize the price of the stock to 1 at time 0. We indicate the difference between the spot price and the strike price of the option by a percentage a in the money relative to the spot price so that the strike price in terms of the percentage of the spot price is $1 - a$. We consider $a < 1$ since $a \geq 1$ represents a strike equal to or less than 0. By allowing a to take all the values less than 1, we can handle options in the money, at the money, and out of the money in a unified framework. Clearly $a = 0$ corresponds to at the money options. When $a < 0$ the call option is, in fact, out of the money. For example, suppose the spot price of a stock is \$80. Consider two call options with strike prices of \$60 and \$100, respectively. Then the first one is $(80 - 60)/80 = 25\%$ in the money and second is $(80 - 100)/80 = -25\%$ in the money (or 25% out of the money). We use p to denote the percentage premium of the call option relative to the spot price. We will assume $p > a^+$ so that there is no arbitrage opportunity. Given an investment system x, replacing each trade in x with a corresponding trade of buying or writing a call option a in the money with a premium p yields new investment systems characterized by

$$c(x, a, p) = \frac{(x + a)^+ - p}{p}$$

and

$$w(x, a, p) = \frac{p - a - (x + a)^-}{1 - p},$$

respectively.

Our problem becomes one of comparing investment systems $c(x, a, p)$ and $w(x, a, p)$ with x. Two factors pertinent to the investment theory and practice are crucial. First, to evaluate an investment system, one has to consider both its return and its risk. For investment systems with the same return, the smaller the risk the better. This is usually formulated in a mathematical form by using a concave utility function. Second, when actually using an investment system in practice, one has to have a matching money management strategy to decide how large a portion of the total investment capital is put into the investment system. Thus, we will use the expected utility (with respect to a general utility function) of an investment system under the best money management strategy to compare different investment systems. In mathematical form, for a general utility function u, given an investment system x, we define

$$K(x) := \sup\{\mathcal{E}[u(1+sx)] \mid s \in [0,+\infty)\},$$

the maximum utility under best money management. We will call $K(x)$ the effectiveness measure for the investment system x. When $u(t) = \ln(t)$ for $t > 0$ and $u(t) = -\infty$ for $t \leq 0$, $K(x)$ becomes the well known Kelly criterion [16]. It is the compounded expected cumulative return in log scale under the best investment capital allocation. In other words, $\exp(K(x))$ is the compounded expected cumulative return of investment system x under the best investment size. Thus, the larger the K value, the better the investment system. The Kelly criterion corresponds to an aggressive money management strategy that yields the best expected compounded return compared to any other money management methods. We refer to [19, 26, 27, 29] for applications and properties of the Kelly criterion. However, it is also known that the optimal investment size under the Kelly criteria may yield much higher risk than most investors can tolerate. Thus, it is useful to consider effective measures generated by other utility functions. It is still true that a larger value of $K(x)$ indicates a better investment system x although the goal may no longer be maximizing the expected cumulative return. The scalar s is the percent of the capital allocated to the investment system (with the reminder being cash). Normally, as a percentage of the available capital, the range of s should be $[0, 1]$. However, to explore the full potential of the investment system, here we allow s to take all the values in $[0, +\infty)$, with $s > 1$ representing trading on margins (with a 0 margin interest rate). While a general utility function allows us the freedom to model different levels of tolerance to risk, we must impose a certain restrictions on the properties of such utility function for relevance to financial modeling and tractability. We assume that the utility function u is upper semicontinuous and satisfying the following properties:

(u1) (Risk aversion) u is strictly concave,

(u2) (Profit seeking) u is strictly increasing, and $\lim_{t\to+\infty} u(t) = +\infty$

(u3) (Bankruptcy forbidden) For any $t < 0$, $u(t) = -\infty$ and $\lim_{t\to 0+} u(t) = -\infty$,

(u4) (Standardized) $u(1) = 0$ and u is differentiable at $t = 1$.

For example, $u(t) := t - t^{-r}$ for $r > 0$ is a class of utility functions having the above properties. The parameter r allows investors to select preferred level of risk aversion. Property (u4) is not restrictive because one can always shift the utility function to satisfy this condition.

We summarize some of the properties of K in the next theorem, which will be used later.

Theorem 3.1. *The indicator $K(x)$ for the effectiveness of an investment system x has the following properties:*

(i) *(Nonnegativity) $K(x) \geq 0$,*

(ii) *(Characterizing arbitrage) $K(x) = +\infty$ if and only if $x \geq 0$ and $x \neq 0$,*

(iii) *(Characterizing profitable system) $K(x) = 0$ if and only if $\mathcal{E}[x] \leq 0$,*

(iv) *(Scaling invariance) for any $r > 0$, $K(rx) = K(x)$.*

Proof. Since $\mathcal{E}[u(1+sx)] = 0$ when $s = 0$ (if one does not invest then there is neither return nor risk), $K(x)$ is always nonnegative as stated in (i).

Note that if $x \geq 0$, then all trades end up in profit. Of course, under our idealized environment of a zero margin interest rate, one should buy on margins as much as one can and, therefore, $K(x) = +\infty$. Viewing x as a vector in R^N, we use $x^{\downarrow}$ to denote the vector derived from x by rearranging its components in nonincreasing order. When the components of x can take both positive and negative values, we have $x_1^{\downarrow} > 0 > x_N^{\downarrow}$. In this case, $f_x(s) := \mathcal{E}[u(1+sx)]$ is a strictly concave function of s defined on $[0, -1/x_N^{\downarrow})$. Since $f_x(0) = 0$ and $\lim_{s\to -1/x_N^{\downarrow}} f_x(s) = -\infty$, f_x attains its maximum at a unique $\bar{s} \in [0, -1/x_N^{\downarrow})$. Thus, $K(x) = f_x(\bar{s}) < +\infty$. This proves (ii).

We now turn to property (iii). By Jenson's inequality and (u1) for any $s \in [0, +\infty)$ such that $\mathcal{E}[u(1+sx)] > -\infty$ we have

$$\mathcal{E}[u(1+sx)] \leq u(1+s\mathcal{E}[x]).$$

It follows from (u2) and (u4) that, for any investment system x with a non-positive expected return, $K(x) = 0$. On the other hand, if $\bar{s} = 0$ then

$$0 \geq \lim_{s \to 0+} \frac{f_x(s) - f_x(0)}{s} = \mathcal{E}[u'(1)x] = u'(1)\mathcal{E}[x].$$

By property (u2), $u'(1) > 0$ and, therefore, $\mathcal{E}[x] \leq 0$. This validates (iii).

Property (iv) follows directly from $f_x(s) = f_{rx}(s/r)$, for any $r > 0$. Q.E.D.

4 Option pricing intervals

Throughout this section, x is an investment system with $\mathcal{E}[x] \geq 0$. We now compare the three investment systems x, $c(x,a,p) = \frac{(x+a)^+ - p}{p}$ and $w(x,a,p) = \frac{p-a-(x+a)^-}{1-p}$. using the effectiveness measure K. Let us start with two extreme cases. When $a \geq (x_N^{\downarrow})^-$, the investment system of writing covered calls $w(x,a,p)$ contains identical elements $\frac{p-a}{1-p} > 0$ and

$$c(x,a,p) = \frac{x+a-p}{p}.$$

Thus, $K(w(x,a,p)) = +\infty$ and $K(c(x,a,p)) = K(x+a-p) \leq K(x)$. In other words, $w(x,a,p)$ provides an arbitrage opportunity while $c(x,a,p)$ is inferior to x. Similarly, when $a \leq -(x_1^{\downarrow})^+$, the investment system of buying calls $c(x,a,p)$ contains identical elements -1 and

$$w(x,a,p) = \frac{x+p}{p}.$$

Thus, $K(w(x,a,p)) = K(x+p) \geq K(x)$, which is to say $c(x,a,p)$ always loses money while $w(x,a,p)$ is always superior to x. This shows that deeply in the money or out of the money calls are in general unfavorable to buyers and favorable to covered call writers.

In what follows, we assume that $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ is fixed. Since $K(c(x,a,p))$ and $K(w(x,a,p))$ are functions of p, the relative merit of these investment systems are determined by p, the percentage premium for the option. We start by identifying the values of p such that the expectations of $c(x,a,p)$ or $w(x,a,p)$ becomes less than 0.

Theorem 4.1. *Let x be an investment system and let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$. Define*

$$\begin{aligned} u_a &= \mathcal{E}[(x+a)^+] \\ l_a &= a + \mathcal{E}[(x+a)^-]. \end{aligned} \tag{4.1}$$

Then, for $p \geq u_a$, $K(c(x,a,p)) = 0$ and, for $p \leq l_a$, $K(w(x,a,p)) = 0$. Moreover,

$$u_a - l_a = \mathcal{E}[x]. \tag{4.2}$$

Proof. It is easy to calculate that

$$\mathcal{E}[c(x,a,p)] = \frac{\mathcal{E}[(x+a)^+] - p}{p},$$

and

$$\mathcal{E}[w(x,a,p)] = \frac{p - a - \mathcal{E}[(x+a)^-]}{1-p}.$$

Clearly $\mathcal{E}[c(x,a,u_a)] = \mathcal{E}[w(x,a,l_a)] = 0$. The conclusion of the theorem follows immediately from Theorem 3.1. The formula (4.2) follows from a direct computation. Q.E.D.

Next, we show that for any investment system with an appropriate premium for the option, the expected utility of the related investment systems of buying calls and writing covered calls can be balanced.

Theorem 4.2. *For any investment system x and $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$, there exists a unique option premium $p_a^* \in (l_a, u_a)$ such that*

$$K(c(x,a,p_a^*)) = K(w(x,a,p_a^*)).$$

Proof. As the maximum of locally Lipschitz functions both $K(c(x,a,p))$ and $K(w(x,a,p))$ are continuous (in fact, locally Lipschitz) functions of p. Since $K(c(x,a,l_a)) > K(w(x,a,l_a)) = 0$ and $0 = K(c(x,a,u_a)) < K(w(x,a,u_a))$ there exists $p_a^* \in (l_a, u_a)$ such that

$$K(c(x,a,p_a^*)) = K(w(x,a,p_a^*)).$$

When $p \in (l_a, u_a)$, $\mathcal{E}[c(x,a,p)] > 0$ so that the investment system $c(x,a,p)$ makes money. Moreover, for buying calls the lower the premium the better (strictly). Thus, $K(c(x,a,p))$ is a strictly decreasing function on (l_a, u_a). Similarly, $K(w(x,a,p))$ is a strictly increasing function on (l_a, u_a). Thus, the p_a^* identified above is unique. Q.E.D.

Intuitively, the option investment systems of buying calls and writing covered calls are the two sides of a zero-sum game relative to the underlying investment system. Naturally we cannot expect both of them to be better than the original system. In fact, Example 7.1 in Section 7 shows that they can be both inferior to the original investment system, i.e., useful investment information could be lost in the process of trading options. In general we have the following result.

Theorem 4.3. *For any investment system x and $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ and $p \in [l_a, u_a]$,*

$$\min(K(c(x,a,p)), K(w(x,a,p))) \leq K(x).$$

Proof. Let $x = (x(1), \ldots, , x(N))$. Let $\bar{s}_C$ and $\bar{s}_W$ be the best investment sizes of investment systems $c(x,a,p)$ and $w(x,a,p)$ relative to the given utility function, respectively. Then

$$K(c(x,a,p)) = \mathcal{E}\left[u\left(1 + \bar{s}_C \frac{(x(n)+a)^+ - p}{p}\right)\right]$$

and

$$K(w(x,a,p)) = \mathcal{E}\left[u\left(1 + \bar{s}_W \frac{p - a - (x(n)+a)^-}{1-p}\right)\right].$$

Set $\alpha = p/\bar{s}_C$ and $\beta = (1-p)/\bar{s}_W$. Since u is a concave function we have

$$\begin{aligned}
& \frac{\alpha}{\alpha+\beta} K(c(x,a,p)) + \frac{\beta}{\alpha+\beta} K(w(x,a,p)) \\
= \ & \frac{\alpha}{\alpha+\beta} \mathcal{E}[u(1 + \bar{s}_C c(x,a,p))] + \frac{\beta}{\alpha+\beta} \mathcal{E}[u(1 + \bar{s}_W w(x,a,p))] \\
\leq \ & \mathcal{E} u\left[\left(1 + \frac{\alpha}{\alpha+\beta} \bar{s}_C c(x,a,p) + \frac{\beta}{\alpha+\beta} \bar{s}_W w(x,a,p)\right)\right] \\
= \ & \mathcal{E}\left[u\left(1 + \frac{1}{\alpha+\beta}((x+a)^+ - p) + \frac{1}{\alpha+\beta}(p - a - (x+a)^-)\right)\right] \\
= \ & \mathcal{E}\left[u\left(1 + \frac{1}{\alpha+\beta} x\right)\right] \leq K(x).
\end{aligned}$$

Q.E.D.

Corollary 4.4. *Let x be an investment system and let $p_a^* \in (l_a, u_a)$ be the option premium stated in Theorem 4.2. Then*

$$K(c(x, a, p_a^*)) = K(w(x, a, p_a^*)) \le K(x).$$

Note that when $p \le l_a$, $w(x, a, p)$ loses money and when $p \ge u_a$, $c(x, a, p)$ loses money. Thus, the only way that the option investment system $c(x, a, p)$ ($w(x, a, p)$) can be better than x is at the expense of $w(x, a, p)$ ($c(x, a, p)$) through a favorable option premium p. Does such an option premium always exist? The answer is affirmative. For a outside the interval $(-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ we already observed this from our discussion on the extreme cases. The next theorem is about the nontrivial case when $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$.

Theorem 4.5. *Let x be an investment system and let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$. Then there exists a unique option premium p_a^W and a unique option premium p_a^C such that*

$$K(c(x, a, p_a^C)) = K(w(x, a, p_a^W)) = K(x).$$

Consequently, when $p < p_a^C$, $K(c(x, a, p)) > K(x)$ and when $p > p_a^W$, $K(w(x, a, p)) > K(x)$.

Proof. When p approaches 0, for any trade n in the investment system with $x(n) + a > 0$, the percentage gain of the corresponding trade in $c(x, a, p)$, $(x(n) + a)/p - 1$ approaches to $+\infty$ and for any trade n in the investment system with $x(n) + a \le 0$ the percentage loss of the corresponding trade in $c(x, a, p)$ is always -1. Thus, $K(c(x, a, p))$ approaches $+\infty$ as p approaches 0. Since $K(c(x, a, p))$ is a strictly decreasing function and $K(c(x, a, p_a^*)) \le K(x)$, there exists a unique $p_a^C \in (0, p_a^*]$ such that $K(c(x, a, p_a^C)) = K(x)$.

Similarly, when $p = (x_N^{\downarrow})^-$, every entry in

$$w(x, a, p) = ((p - a - (x(1) + a)^-)/(1 - p), \ldots, (p - a - (x(N) + a)^-)/(1 - p))$$

is nonnegative. Thus, $K(w(x, a, p))$ approaches $+\infty$ as p approaches $(x_N^{\downarrow})^-$. Since $K(w(x, a, p))$ is a strictly increasing function and $K(w(x, a, p_a^*)) \le K(x)$, there exists a unique $p_a^W \in [p_a^*, (x_N^{\downarrow})^-)$ such that $K(w(x, a, p_a^W)) = K(x)$. Q.E.D.

While p_a^C and p_a^W are critical thresholds for $c(x, a, p)$ and $w(x, a, p)$ to outperform the original investment system x, we cannot expect explicit formulae for them in general. Thus, they can only be calculated numerically and finding estimates for p_a^C and p_a^W is useful to narrow down a range for the numerical search. In the nest section, We will show that l_a and $\bar{p}_a$ are lower and upper bounds for p_a^C and p_a^W, respectively, using majorization.

5 Majorization and bounds for p^C and p^W

It turns out a crucial tool for gaining insight and deriving bounds of the critical thresholds p_a^C and p_a^W is majorization (see [1, 3]). First let us recall the definition. Let $x, y \in R^N$. We say y majorized x, denoted $x \prec y$ provided that

$$\sum_{n=1}^{k} x_n^{\downarrow} \leq \sum_{n=1}^{k} y_n^{\downarrow}, \text{ for } k = 1, \ldots, N-1,$$

and

$$\sum_{n=1}^{N} x_n^{\downarrow} = \sum_{n=1}^{N} y_n^{\downarrow}.$$

The most important property of majorization relevant to our discussion here is given in the following theorem.

Theorem 5.1. [3, Theorem II.3.1] *Let x and y be two vectors in R^N. Then $x \prec y$ if and only if $\sum_{n=1}^{N} f(y_n) \leq \sum_{n=1}^{N} f(x_n)$ for all concave functions $f : R \to R$.*

Here is the form we will use this theorem.

Corollary 5.2. *Let x be an investment systems and let y be an investment system derived by replacing trades in x by corresponding options. Suppose that $x \prec y$ ($y \prec x$). Then $K(x) \geq K(y)$ ($K(x) \leq K(y)$).*

Proof. Recall that $K(x) = \sup\{f_x(s) \mid s \in [0, \infty)\}$, where $f_x(s) = \mathcal{E}[u(1 + sx)]$ is strictly concave. Thus, if $x \prec y$ ($y \prec x$) then, for all s in the domains of f_x and f_y,

$$f_x(s) \geq f_y(s) \ (f_x(s) \leq f_y(s)).$$

It remains to take maximum over s. Q.E.D.

Since $K(w(x, a, l_a)) = 0$, trading option at premium l_a the writer of the covered call will lose money to the buyer of the call. Thus, we have reason to expect l_a to be a lower bound for p_a^C. However, we need to make a detour to prove this fact. Let x be an investment system. Let $c(x, a, p) = \frac{(x+a)^+ - p}{p}$ be the corresponding investment system of buying calls at premium p. Consider the investment system $pc(x, a, p) = (x + a)^+ - p$, which has the following meaning: for each trade in x, dividing the capital (which we again standardize to 1) into two parts of $1-p$ and p the investment system uses p to purchase a corresponding call option a in the money with a premium p and reserves the remaining $1-p$ in cash. We will call $pc(x,a,p)$ the investment system of buying option with full capital reserve corresponding to x and use it as a bridge to reach our anticipated conclusion that l_a is a lower bound for p_a^C. The key observation is:

Theorem 5.3. *Let x be an investment system, let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ and let $pc(x,a,p)$ be the investment system of buying option with full capital reserve corresponding to x. Let $\underline{p}$ be an option premium such that*

$$\mathcal{E}[x] = \mathcal{E}[\underline{p}c(x,a,\underline{p})].$$

Then,

$$\underline{p}c(x,a,\underline{p}) \prec x.$$

Proof. Assume without loss of generality that $x = x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_N^{\downarrow})$ and that

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots, \geq x_k^{\downarrow} \geq -a > x_{k+1}^{\downarrow} \geq \ldots \geq x_N^{\downarrow}.$$

Since $x \to ((x+a)^+ - p)$ is increasing, $\underline{p}c(x,a,\underline{p}) = \underline{p}c(x,a,\underline{p})^{\downarrow} = (c_1^{\downarrow}, c_2^{\downarrow}, \ldots, c_N^{\downarrow})$ where

$$c_n^{\downarrow} = \begin{cases} x_n^{\downarrow} + a - \underline{p} & n = 1, \ldots, k \\ -\underline{p} & n = k+1, \ldots, N. \end{cases}$$

It follows that

$$h_n := x_n^{\downarrow} - c_n^{\downarrow} = \begin{cases} \underline{p} - a & n = 1, \ldots, k \\ x_n^{\downarrow} + \underline{p} & n = k+1, \ldots, N. \end{cases}$$

The condition $\mathcal{E}[x] = \mathcal{E}[\underline{p}c(x,a,\underline{p})]$ implies that $\underline{p} = a + \mathcal{E}[(x+a)^-] \geq a$ and

$$\begin{aligned} \sum_{n=1}^{N} h_n &= \sum_{n=1}^{N} x_n^{\downarrow} - \sum_{n=1}^{N} c_n^{\downarrow} = \sum_{n=1}^{N} x_n - \sum_{n=1}^{N} c_n \\ &= N\left(\mathcal{E}[x] - \mathcal{E}[\underline{p}c(x,a,\underline{p})]\right) = 0. \end{aligned} \tag{5.1}$$

Clearly, h_n is nonincreasing. Since $h_N \geq 0$ implies $h_n = 0$ for all $n = 1, 2, \ldots, N$ and $x = \underline{p}c(x,a,\underline{p})$ we consider the nontrivial case when $h_N < 0$.

Let m be the smallest integer such that $h_m < 0$. Then $h_n < 0$ for $n = m, m+1, \ldots, N$. It follows from $\sum_{n=1}^{N} h_n = 0$ that, for $k = m, m+1, \ldots, N$,

$$\sum_{n=1}^{k} h_n \geq 0. \tag{5.2}$$

Since $h_n \geq 0$ for $n = 1, 2, \ldots, m-1$ inequalities (5.2) holds for all $k = 1, 2, \ldots, N$, that is, $\underline{p}c(x, a, \underline{p}) \prec x$. Q.E.D.

Theorem 5.4. *Let x be an investment system and let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$. Then*

$$p_a^C \geq l_a.$$

Proof. Solving equation

$$\mathcal{E}[\underline{p}c(x, a, \underline{p})] = \mathcal{E}[x]$$

we have $\underline{p} = l_a$. It follows from Theorem 5.3 that, for any $s \geq 0$,

$$f_x(s) \leq f_{l_a c(x,a,l_a)}(s).$$

Thus, $K(x) \leq K(l_a c(x, a, l_a)) = K(c(x, a, l_a))$ by the scaling invariant property (iv) of K in Theorem 3.1. Since $K(c(x, a, p_a^*)) \leq K(x)$ we have $p_a^C \in [l_a, p_a^*]$. This establishes l_a as a lower bound for p_a^C. Q.E.D.

Similarly we can expect u_a to be an upper bound for p_a^W. This is true but we can do better in showing that $\bar{p}_a := u_a/(1+\mathcal{E}[x]) \leq u_a$ is an upper bound for p_a^W. First, we record that $\bar{p}_a$ is the value at which the expected returns of all the three investment systems x, $c(x, a, \bar{p}_a)$ and $w(x, a, \bar{p}_a)$ coincide.

Lemma 5.5. *Let x be an investment system and let $a < 1$. Define $\bar{p}_a := u_a/(1 + \mathcal{E}[x])$. Then*

$$\mathcal{E}[x] = \mathcal{E}[c(x, a, \bar{p}_a)] = \mathcal{E}[w(x, a, \bar{p}_a)]$$

Proof. This follows from a simple direct computation. Q.E.D.

Theorem 5.6. *Let x be an investment system, let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ and let $\bar{p}_a := u_a/(1 + \mathcal{E}[x]) = \mathcal{E}[(x + a)^+]/(1 + \mathcal{E}[x])$. Then*

$$p_a^W \leq \bar{p}_a.$$

Proof. We need to show that $K(x) \leq K(w(x,a,\bar{p}_a))$. This follows from the stronger result in our next theorem. Q.E.D.

We now show that for the option premium $\bar{p}_a$ that balances the expected returns of the investment systems x and $w(x,a,\bar{p}_a)$, $w(x,a,\bar{p}_a)$ is majorized by x which is in turn majorized by $c(x,a,\bar{p}_a)$.

Theorem 5.7. *Let x be an investment system, let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$ and let $\bar{p}_a$ be the option premium such that $\mathcal{E}[x] = \mathcal{E}[w(x,a,\bar{p}_a)]$. Then*

$$w(x,a,\bar{p}_a) \prec x \prec c(x,a,\bar{p}_a).$$

Consequently,

$$K(c(x,a,\bar{p}_a)) \leq K(x) \leq K(w(x,a,\bar{p}_a)).$$

Proof. Assume without loss of generality that $x = x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_N^{\downarrow})$ and that

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_j^{\downarrow} \geq \frac{\bar{p}_a - a}{1 - \bar{p}_a} > x_{j+1}^{\downarrow} \geq \ldots, \geq x_k^{\downarrow} \geq -a > x_{k+1}^{\downarrow} \geq \ldots \geq x_N^{\downarrow}.$$

Since $x \to (\bar{p}_a - a - (x+a)^-)/1 - \bar{p}_a$ is increasing, $w(x,a,\bar{p}_a) = w(x,a,\bar{p}_a)^{\downarrow} = (w_1^{\downarrow}, w_2^{\downarrow}, \ldots, w_N^{\downarrow})$ where

$$w_n^{\downarrow} = \begin{cases} \frac{\bar{p}_a - a}{1-\bar{p}_a} & n = 1, \ldots, k \\ \frac{\bar{p}_a + x_n^{\downarrow}}{1-\bar{p}_a} & n = k+1, \ldots, N. \end{cases}$$

Similarly, we have $c(x,a,\bar{p}_a) = c(x,a,\bar{p}_a)^{\downarrow} = (c_1^{\downarrow}, c_2^{\downarrow}, \ldots, c_N^{\downarrow})$ where

$$c_n^{\downarrow} = \begin{cases} \frac{x_n^{\downarrow} + a - \bar{p}_a}{\bar{p}_a} & n = 1, \ldots, k \\ -1 & n = k+1, \ldots, N. \end{cases}$$

For $n = 1, 2, \ldots, N$, defining $a_n = c_n^{\downarrow} - x_n^{\downarrow}$ and $b_n = x_n^{\downarrow} - w_n^{\downarrow}$ we have

$$a_n = \begin{cases} \frac{a - \bar{p}_a + x_n^{\downarrow}(1-\bar{p}_a)}{\bar{p}_a} & n = 1, \ldots, k \\ -(1 + x_n^{\downarrow}) & n = k+1, \ldots, N. \end{cases}$$

and

$$b_n = \begin{cases} \frac{a - \bar{p}_a + x_n^{\downarrow}(1-\bar{p}_a)}{1-\bar{p}_a} & n = 1, \ldots, k \\ \frac{-\bar{p}_a(1 + x_n^{\downarrow})}{1-\bar{p}_a} & n = k+1, \ldots, N. \end{cases}$$

It is easy to see that $b_1 \geq b_2 \geq \ldots \geq b_j \geq 0$ and $b_n < 0$ for $n = j+1, \ldots, N$. Similar to (5.1), the equation $\mathcal{E}[x] = \mathcal{E}[w(x, a, \bar{p}_a)]$ implies that $\sum_{n=1}^{N} b_n = 0$. Consequently, $\sum_{n=1}^{k} b_n \geq 0$ for $k = 1, 2, \ldots, N-1$. That is to say $w(x, a, \bar{p}_a) \prec x$. Since $a_n = (1 - \bar{p}_a) b_n / \bar{p}_a$, we also have $\sum_{n=1}^{k} a_n \geq 0$ for $k = 1, 2, \ldots, N-1$ and $\sum_{n=1}^{N} a_n = 0$, and, therefore, $x \prec c(x, a, \bar{p}_a)$. Q.E.D.

To summarize we have

Theorem 5.8. *Let x be an investment system with a positive expected return and let $a \in (-(x_1^{\downarrow})^+, (x_N^{\downarrow})^-)$. Then there exists three critical option premiums p_a^C, p_a^W and p_a^* satisfying*

$$l_a \leq p_a^C \leq p_a^* \leq p_a^W \leq \bar{p}_a \leq u_a, \tag{5.3}$$

where $l_a = a + \mathcal{E}[(x+a)^-]$ and $u_a = \mathcal{E}[(x+a)^+]$. They have the following properties:

(a) *when $p \in [p_a^C, p_a^W]$,*

$$\max(K(w(x, a, p)), K(c(x, a, p))) \leq K(x).$$

In particular,

$$K(c(x, a, p_a^*)) = K(w(c, a, p_a^*)) \leq K(x).$$

(b) *when $p \in [l_a, p_C]$,*

$$K(c(x, a, p)) \geq K(x) \geq K(w(x, a, p)).$$

(c) *when $p \in [p_W, u_a]$,*

$$K(w(x, a, p)) \geq K(x) \geq K(c(x, a, p)).$$

Properties of these investment systems outside the interval $[l_a, u_a]$ are less important. We record them below for completeness. The justifications for these facts have already appeared in the discussions.

Theorem 5.9. *Let x be an investment system with a positive expected return and let $a < 1$. Then*

(a) *when $p \geq u_a$, the investment system $w(x, a, p)$ improves x while the investment system $c(x, a, p)$ loses money, and*

(b) *when $p \leq l_a$, the investment system $c(x, a, p)$ improves x while the investment system $w(x, a, p)$ loses money.*

6 A robust option replacement trading strategy

Consider a stock investment system x. Let us examine options of the stock a in the money. Theorem 5.8 suggests the following option replacement trading strategy: if the call option premium $p > p_a^W$ write covered calls, if $p < p_a^C$ buy calls and otherwise buy stocks. The percentage capital s put into the investment positions should be the one that maximizing the expected utility. Thus, the effectiveness measure for this investment system is

$$m(x) = \max(K(x), K(c(x, a, p), K(w(x, a, p)).$$

The motivating example leads to the intuition that the option replacement trading strategy is robust with respect to perturbation. We now make this precise. The key is the following result.

Theorem 6.1. *For any utility function satisfying assumptions (u1)–(u4), the corresponding effectiveness measure K for investment systems is locally Lipschitz near every non-arbitrage profitable investment system. In other words, K is locally Lipschitz near every x satisfying $K(x) \in (0, +\infty)$.*

Proof. Suppose that x is a non-arbitrage profitable investment system so that $K(x) \in (0, +\infty)$. Then $0 \in (-1/x_1^{\downarrow}, -1/x_N^{\downarrow})$. Let $s(x)$ be the best investment size corresponding to x. We have

$$K(x) = \mathcal{E}[u(1 + s(x)x)]$$

and $s(x) \in (0, -1/x_N^{\downarrow})$. Clearly, for all $n = 1, \ldots, N$, $1+s(x)x_n \in \operatorname{int}(\operatorname{dom} u)$. As a concave function u is locally Lipschitz in the interior of its domain. it follows that there exists an $\delta > 0$ such that the function $y \to \mathcal{E}[u(y)]$ is Lipschitz with a Lipschitz constant L on $B_\delta(1 + s(x)x)$. Furthermore we can select $\eta \in (0, -x_N^{\downarrow}/2)$ small enough so that $y \in B_\eta(x)$ implies that $1+s(y)y, 1+s(x)y, 1+s(y)x \in B_\delta(1+s(x)x)$, where $s(y)$ is the best investment size corresponding to y that is $K(y) = \mathcal{E}[u(1 + s(y)y)]$. Now, for any $y \in B_\eta(x)$, we have

$$\begin{aligned}
Ls(y)\|y - x\| &\geq \mathcal{E}[u(1 + s(y)y)] - \mathcal{E}[u(1 + s(y)x)] \\
&\geq K(y) - K(x) \\
&\geq \mathcal{E}[u(1 + s(x)y)] - \mathcal{E}[u(1 + s(x)x)] \\
&\geq -Ls(x)\|y - x\|.
\end{aligned}$$

Since both $s(x)$ and $s(y)$ belongs to the interval $(0, -2/x_N^{\downarrow}]$, K is a Lipschitz function on $B_\eta(x)$. Q.E.D.

Since both $c(x, a, p)$ and $w(x, a, p)$ are Lipschitz functions of x, it follows that the option replacement trading strategy is robust in the sense that the possible error in the effective measure m caused by a small perturbation around a non-arbitrage profitable investment system x is at most linear with respect to the perturbation. The precise error estimate will depend on the behaviors of the utility function and the underlying investment system. Issues related to concrete implementations will be addressed elsewhere.

7 Examples

In general explicit formulae for $\bar{s}$ and $K(x)$ are not to be expected. However, we can derive them for some special cases. In this section we use the Kelly criterion where the utility function is $u = \ln$ and consider an illustrating example where explicit formula is available.

Let us consider an investment system T whose only possible percentage gains are $\{g, 0, l\}$ with $g > 0 > l$. This is the situation when we can explicitly calculate the efficiency index and we will use it to explore examples to gain some insights. Suppose that the frequency of the percentage gain of a trade to be $g, 0$ and l are ϕ_g, ϕ_0 and ϕ_l, respectively. Then by Theorem 3.2 in [29] we have

$$K(T) = \phi_g \ln \frac{\phi_g(l-g)}{(\phi_g + \phi_l)l} + \phi_l \ln \frac{\phi_l(g-l)}{(\phi_g + \phi_l)g}. \tag{7.1}$$

We use $c(p) = c(T, 0, p)$ and $w(p) = w(T, 0, p)$ to denote the corresponding investment systems of replacing each trade in T by buying an at money call option and by writing an at money covered call option with premium p, respectively. Then the trades in $c(p)$ have only two possible outcomes $g/p - 1$ and -1 with frequency ϕ_g and $1 - \phi_g$, respectively, and the trades in $w(p)$ also have only two possible outcomes $p/(p-1)$ and $(p+l)/(1-p)$ with frequency $1 - \phi_l$ and ϕ_l, respectively. Thus, again we can use Theorem 3.2 in [29] to explicitly calculate the efficiency index of $c(p)$ and $w(p)$ as functions of the option premium p given below:

$$K(c(p)) = \phi_g \ln \frac{\phi_g g}{p} + (1 - \phi_g) \ln \frac{(1-\phi_g)g}{g-p}$$

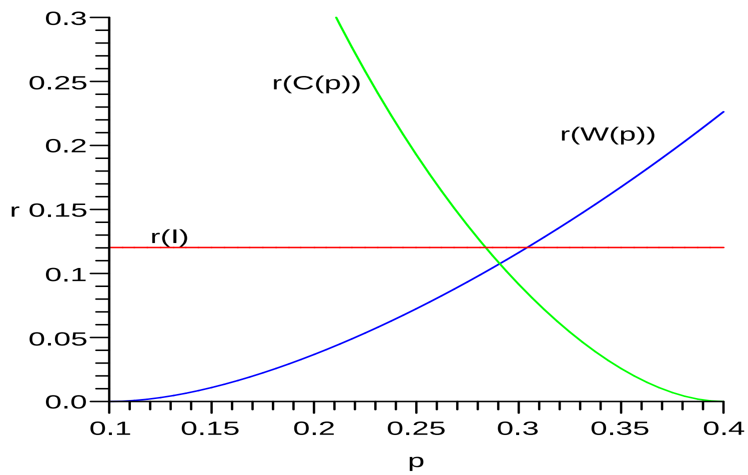

Figure 1: Critical option premiums

and

$$K(w(p)) = (1-\phi_l)\ln\frac{(1-\phi_l)l}{p+l} + \phi_l \ln\frac{-\phi_l l}{p}.$$

Let us examine a concrete example.

Example 7.1. Consider the investment system

$$T = \{0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0, -1\}.$$

We can explicitly calculate that

$$K(T) = 0.12,$$

$$K(c(p)) = 0.8\ln\frac{0.4}{p} + 0.2\ln\frac{0.1}{0.5-p},$$

and

$$K(w(p)) = 0.9\ln\frac{-0.9}{p-1} + 0.1\ln\frac{0.1}{p}.$$

Graphing them together we get Figure 1.

We observe that $K(c(0.4)) = 0$ and $K(w(0.1)) = 0$ gives the values 0.4 and 0.1 where the investment system of buying calls and writing covered calls lose money, respectively. We further observe that 0.1 coincides with the average loss

$$\mathcal{E}[T^-] = \frac{1}{10} = 0.1$$

and that 0.4 coincides with the average gain

$$\mathcal{E}[T^+] = \frac{8 \cdot 0.5}{10} = 0.4$$

The graphs of $K(c(p))$ and $K(w(p))$ intersect in between $[0.1, 0.4]$ at $p^* = 0.291$. This is the point at which the two investment systems are equally good according to the effective measure. However, the common value $K(c(p^*)) = K(w(p^*)) = 0.108$ is smaller than $K(T) = 0.12$, indicating that some useful investment information has been lost due to the option trading. In fact, this is true for any option premium p in between $p^C = 0.284$ and $p^W = 0.304$ where the graphs of $K(c(p))$ and $K(w(p))$ intersect the horizontal line $K = K(T)$, respectively. It is clear from the picture that when $p < p^C$ ($p > p^W$), $c(p)$ ($w(p)$) improves T and when p belongs to the interval $[p^C, p^W]$ the original investment system T is the best, as concluded in Theorem 5.8. The upper bound $\bar{p}$ satisfies

$$\mathcal{E}[T] = \mathcal{E}[c(\bar{p})] = \mathcal{E}[w(\bar{p})]$$

is not shown in the graph. However,we can check $\bar{p} = 0.308 > 0.304 = p^W$.

One may wonder whether there are situations in which $p_a^W = p_a^C$. It turns out that this is always true in a complete market as discussed in the next section.

8 Relationship with pricing by a replicating portfolio

When the payoff of the option can be exactly replicated by a portfolio of stocks and bonds, pricing according to the no arbitrage principle one can derive the valuation of the option by either directly calculating the cost of the replicating portfolio following the idea of [4, 20] or use the risk neutrality argument in [9]. The option pricing interval $[p_a^C, p_a^W]$ is compatible to both methods in the sense that it will yield the same price for the option. However, we emphasize that the implication for trading is different. The pricing in [4, 9, 20] implies that one should use the replicating portfolio to take the arbitrage opportunity should the actual option value deviated from the theoretical value. This is instable with respect to model inaccuracy as the example in Section 2 shows. On the other hand the option pricing interval method discussed here calls for more stable simple pure positions of stocks, call options or write call options. We now elaborate the compatibility alluded to above.

Given an investment system x, the random variable representing the value of the stock at time 1 is $1+x$ and the random variable representing the payoffs of a call option a in the money is $(x+a)^+$. Suppose that the payoff of the call option can be replicated by a portfolio of α stocks and β bonds in our one period economy. Then,

$$\alpha(1+x)+\beta=(x+a)^+,$$

and the pricing of the option by this replicating portfolio is $p_r=\alpha+\beta$. We can easily verify that

$$c(x,a,p_r)=\frac{(x+a)^+-p_r}{p_r}=\frac{\alpha(1+x)+\beta-p_r}{p_r}=\frac{\alpha}{p_r}x$$

and

$$\begin{aligned} w(x,a,p_r) &= \frac{p_r-a-(x+a)^-}{1-p_r}=\frac{p_r-a+(x+a)-(x+a)^+}{1-p_r} \\ &= \frac{p_r-a+(x+a)-\alpha(1+x)-\beta}{1-p_r}=\frac{1-\alpha}{1-p_r}x. \end{aligned}$$

By the scaling invariance property (iv) of K in Theorem 3.1 we have

$$K(c(x,a,p_r))=K(w(x,a,p_r))=K(x)$$

and, therefore, $p_r=p_a^C=p_a^*=p_a^W$. In other words the thresholds p_a^C and p_a^W both coincide with p_r, the option valuation derived by calculating the cost of the replicating portfolio.

We now turn to the risk neutrality argument in [9]. The idea is that if we can exactly replicate the payoff of the option with a portfolio involving stocks and bonds then the option can be hedged without any risk. Thus, its valuation is the same for all investors regardless of their individual risk tolerance. In particular, we can evaluate the option in an idealized risk neutral economy where investors do not ask for a risk premium for risky investments and, therefore, all investments have the same expected rate of return that equals to that of the risk-less bond. Now in our setting of using the real value, the rate of return of the risk-less bond is 0. Applying our result in this risk neutral economy, the value of the stock at time 1 satisfies the risk neutral probability distribution so that $\mathcal{E}[x] = 0$. It follows that $u_a = l_a$ and, therefore, all the critical option premiums in (5.3) coincide. In particular, $p_a^W = p_a^C = u_a = \mathcal{E}[(x+a)^+]$. Note that $\mathcal{E}[(x+a)^+]$ is exactly the option valuation in the risk neutral economy.

9 Conclusion

We proposed a robust option replacement trading strategy that addresses some of the difficulties of using the arbitrage option pricing and trading strategy in practice when there are inevitable model inaccuracies. This also leads to natural pricing intervals for options in incomplete markets. Estimates for those pricing intervals were derived by using vector majorization. The analysis show case the role of convex and set-valued analysis in financial applications.

Issues related to implementation of this strategy involve investment system designing and testing using historical data. They will be discussed elsewhere.

The analysis here is restricted to a one period economy which provides guide to simple implementable option investment strategies. Similar analysis for multiple period economy and, in the limit, continuous economy is an interesting direction for further research which may provide useful insight in both theory and practice.

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