Convex Analysis in Financial Mathematics

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Abstract

Using the language of convex analysis we describe key results in several important areas of finance: portfolio theory, financial derivative trading and pricing and consumption based asset pricing theory. We hope to emphasize the importance of convex analysis in financial mathematics and also bring attention to researchers in convex analysis interesting issues in financial applications.

Key Words. Convex analysis, duality, financial mathematics.

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1 Introduction

Concave utility functions and convex risk measures play crucial roles in economic and financial problems. The use of concave utility function can at least be traced back to Bernoulli when he posed and solved the St. Petersburg wager problem. They have been the prevailing way to characterize rational market participants since. Markowitz used variation as a risk measure in his pioneering work on portfolio theory [30]. This is a quadratic risk measure that played prominent role in subsequent related work such as capital market asset pricing model and Sharpe ratio for evaluating investment performance. Arzner et al [2] proposed the concept of coherent measure base on the practices of risk control for large clearing houses. This was later generalized to convex risk measure in [15, 16, 17, 38]. Moreover, in the general equilibrium theory of economics, convex sets also play key roles in describing the production, consumption and their exchange. The essential roles of these convex objects made convex analysis an indispensable tool in dealing with problem in finance. The purpose of this paper is to highlight the crucial role of convex analysis in financial research by using the convex analysis language to describe key results in several important areas of finance: portfolio theory, financial derivative trading and pricing and consumption based asset pricing theory.

To set the stage we first layout a discrete model for the financial market in Section 2. This largely follows the notation in [37]. The discrete model avoids much technical difficulty associated with the continuous model of the financial markets and allows us to concentrate more on the principles. After describing the model we explain the concept of arbitrage and the no arbitrage principle. This is followed by the important fundamental theorem of asset pricing in which the no arbitrage condition is characterized by the existence of martingale measures (also known as the risk neutral measures). The proof of this theorem relies on the convex separation theorem, which gives us a first taste of the importance of convex analysis tools. Next we discuss how to use utility functions and risk measures to characterize the preference of market agents in Section 3. We layout assumptions that are commonly imposed on utility functions and risk measured and give their financial explanations. It is interesting to see that the no arbitrage principle can also be characterized with a class of utility functions in terms of the utility being finite. Once the preliminary material is in place we turn to discuss a number of key results in several important areas of finance systematically using convex analysis.

Section 4 is about portfolio theory. We discuss Markowitz portfolio theory first. Related capital asset pricing model follows. We show that mathematically both of them are related to quadratic optimization with linear constraints – the simplest form of convex programming and enjoy explicit solutions. Sharpe ratio, an important measure for performances of different investment methods, is then derived as a consequence. Finally, we touch upon the capital growth model in which maximum capital growth is the goal. We illustrate the general pattern and emphasize that the optimal solution to the capital growth model is not stable.

Section 5 deal with the issue of pricing financial derivatives. We use simple models to illustrate the idea of the prevailing Black -Scholes replicating portfolio pricing method [3, 31] and related Cox-Ross [6, 7] risk-neutral pricing method for financial derivatives. A widely held believe about these methods is that they are independent of individual market player's preference and providing a uniformly applicable pricing mechanism for financial derivatives. However, we show that the replicating portfolio pricing method is a special case of portfolio optimization by maximizing a particular kind of concave utility functions and the risk neutral measure is a natural byproduct of solving the dual problem. Thus, using the Black-Scholes option pricing mechanism and the related risk neutral measure pricing method is, in fact, implicitly accepting a utility function along with other assumptions associated with these methods such as infinite leverage and one can use high frequency trading to maintain the replicating portfolio without paying much transaction cost etc... These observations point to necessary cautions when using the Black-Scholes and Cox-Ross pricing methods. More importantly, understanding these methods are, in fact, related to utility optimization naturally leads to the consideration of their sensitivity. It turns out these pricing methods are rather sensitive to model perturbations: a small deviation from the perceive the market model may well leads to the perceived arbitrage position constructed according to the theoretical market model to become pure losing positions. These theoretical flaw were also reflected in the real markets through the financial crises caused by the collapse of Long Term Capital Management in 1998 (see [33]) and the recent financial crisis of 2008. The unsatisfactory effect of the prevailing pricing and trading mechanism for financial derivatives calls for alternative ways of pricing and trading financial derivatives. It seems that the time tested utility optimization method is still highly relevant and a financial derivative market in which different players using different approaches is reasonable. One of such method emphasizing the robustness of the pricing and trading is discussed with tests conducted using real historical market data. Convex analysis plays a crucial role in this robust pricing and trading method.

In Section 6 we discuss a consumption based pricing model in which the pricing of financial assets are directly determined through the interaction of production, consumption and saving in a competitive market. While the idea of competitive market determines the price can be traced back to Adam Smith's invisible hand, it was L. Walras who first attempted a mathematical model for a general equilibrium in his 1877 treaties "Elements of Pure Economics". Rigorous formulation of the model and the proof of the existence of an equilibrium pricing was achieved by Kenneth Arrow and Gerald Debrue in the 1950's [21]. We choose to present the influential Lucas' model [29]. In analyzing this model, convex analysis is combined with dynamical programming. We also briefly discuss the recent developments on extending the Lucas model to model term structure of interest rates.

This is an attempt to illustrate the importance of convex analysis in financial problems. We selected examples using one period, multi period and dynamic programming models to emphasize that tools in convex analysis, in particular, convex duality is indispensable in dealing with financial problems with different degrees of sophistication. We also hope this will draw attentions of researchers in convex analysis to the many challenges arise in financial applications.

2 A discrete model for the financial markets

We will use a finite set Ω to represent all possible economic states and assume that the natural probability of each state is described by a probability measure P on the power set of Ω. We assume that $P(\omega) > 0$ for all $\omega \in \Omega$. Let $RV(\Omega)$ be the finite dimensional Hilbert Space of all random variables defined on Ω, with inner product

$$\langle \xi, \eta \rangle = \mathbf{E}^P(\xi\eta) = \int_\Omega \xi\eta dP = \sum_{\omega\in\Omega} \xi(\omega)\eta(\omega)P(\omega).$$

When there is no risk of ambiguity we will omit the superscript P and write $\mathbf{E}(\xi\eta) = \mathbf{E}^P(\xi\eta)$. For $\xi \in RV(\Omega)$ we use $\xi > 0$ to signal $\xi(\omega) \geq 0$ for all $\omega \in \Omega$ and at least one of the inequality is strict. We consider a discrete model in which trading action can only take place at $t = 0, 1, 2, \ldots$.

Let $\mathcal{F} = \{\mathcal{F}_t \mid t = 0, 1, \ldots\}$ be an *information system* of σ-algebras of subsets of Ω, that is,

$$\sigma(\{\Omega\}) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_1 \subset \ldots \text{ and } \cup_{t=0}^{\infty} \mathcal{F}_t = \sigma(\Omega).$$

Here, for each $t = 0, 1, \ldots$, $\mathcal{F}_t$, represents available information at time t. Thus, an information system represents a framework in which we never loss any information. Moreover, to begin with at $t = 0$ we know nothing and our knowledge increase with time t. We will often consider finite period economy where $t = 0, 1, \ldots, T$. In this case $\mathcal{F}_T = \sigma(\Omega)$. The triple $(\Omega, \mathcal{F}, P)$ is a way to model the gradually available information about the economy.

Proposition 2.1. *In a finite period economy. for each $t = 0, 1, \ldots, T$, let P_t be the set of atoms of the σ-algebra $\mathcal{F}_t$. Then*

(a) for each t, P_t is a partition of Ω, and

(b) $\mathcal{F}$ is an information system if and only if

$$\{\Omega\} = P_0 \prec P_1 \prec \cdots \prec P_T = \{\{\omega\} : \omega \in \Omega\}.$$

Here $P \prec Q$ signifies that Q is a refinement of P.

Let $\mathcal{A} = \{a^0, a^1, \ldots, a^M\}$ be $M + 1$ assets in which a^0 is risk free and the rest are risky. For example, we can think of a_0 as the US treasury bonds that carries a fixed interest and $a^1, \ldots, a^M$ as various stocks or stock indices. We use a vector stochastic process $S := \{S_t\}_{t=0,1,\ldots}$ to represent the prices of these assets, where $S_t := (S_t^0, S_t^1, \ldots, S_t^M)$ is the *discounted price vector* of the $M + 1$ assets at time t with S_t^m representing the discounted price of asset a^m at t. Using the discounted price, we have $S_t^0 = 1$ for all $t = 0, 1, \ldots$. Thus, we can always think a^0 as cash. Since at time t the information available is represented by the σ-algebra $\mathcal{F}_t$, it is reasonable to assume S_t is $\mathcal{F}_t$-measurable. If $S := \{S_t\}_{t=0,1,\ldots}$ is a stochastic process such that, for every t, S_t is $\mathcal{F}_t$-measurable then we say S is $\mathcal{F}$-adapted. We will refer $\mathcal{A}$ described above as a financial market model.

Definition 2.2. (Portfolio) *A* portfolio Θ_t *on the time interval* $[t-1, t)$ *is an $\mathcal{F}_{t-1}$ measurable random vector $\Theta_t = (\Theta_t^0, \Theta_t^1, \ldots, \Theta_t^M)$ where Θ_t^m indicates the number of shares of asset a^m in the portfolio. We assume that such a portfolio is always purchased in the beginning of the interval and liquidated at the end of the interval. It is clear that, the* acquisition price *and the* liquidation price *of the portfolio Θ_t are $\Theta_t \cdot S_{t-1}$ and $\Theta_t \cdot S_t$, respectively.*

Since the decision of purchasing portfolio Θ_t has to be made at $t-1$, Θ_t should be $\mathcal{F}_{t-1}$-measurable to be compatible with the information available. A stochastic process $\Theta = (\Theta_1, \Theta_2, \ldots)$ is said to be $\mathcal{F}$-predictable if, for all t, Θ_t is $\mathcal{F}_{t-1}$-measurable.

Definition 2.3. (Trading strategy) *A* trading strategy *is an $\mathcal{F}$-predictable process of portfolios $\Theta = (\Theta_1, \Theta_2, \ldots)$, where Θ_t denotes the portfolio in the time interval $[t-1, t)$. A trading strategy is self-financing if at any t the acquisition price and the liquidation price of the portfolios are the same, that is to say,*

$$\Theta_t \cdot S_t = \Theta_{t+1} \cdot S_t, \forall t = 1, 2, \ldots.$$

Given a trading strategy Θ and an initial wealth w_0. We have, at time $t = 0$,

$$\Theta_1 \cdot S_0 = w_0. \tag{2.1}$$

The net wealth at time $t = T$ is

$$w_T = \sum_{t=1}^{T} \Theta_t \cdot (S_t - S_{t-1}) + w_0. \tag{2.2}$$

Clearly,

$$G_T(\Theta) = \sum_{t=1}^{T} \Theta_t \cdot (S_t - S_{t-1}) = w_T - w_0 \tag{2.3}$$

is the net gain.

With a little thinking we will realize that as far as the net gain is concerned for every trading strategy we can come up with an equivalent self-financing trading strategy: we only need to absorb all the difference of the acquisition price and liquidation prices into the cash asset. We will use $\mathcal{T}(\mathcal{A})$ to denote the set of all self-financing trading strategies as a vector space with respect to the canonical addition and scalar multiplication. Clearly when we consider a finite period economy $\mathcal{T}(\mathcal{A})$ is a finite dimensional vector space.

For any $\Theta \in \mathcal{T}(\mathcal{A})$, we have

$$w_T = \Theta_T \cdot S_T - \Theta_1 \cdot S_0 + w_0 = \Theta_T \cdot S_T, \tag{2.4}$$

and

$$G_T(\Theta) = \Theta_T \cdot S_T - \Theta_1 \cdot S_0. \tag{2.5}$$

Definition 2.4. (Arbitrage) *A* trading strategy *is called an arbitrage if $G_t(\Theta) \geq 0$ for all t and at least one of them is strictly positive.*

Intuitively, an arbitrage trading strategy is a risk free way of making money. In theory if such a strategy exists then everyone will be pursuing it and the opportunity should disappear very quickly. Thus, usually in theoretical analysis one assumes no arbitrage trading strategy exists.

A useful characterization of the principle of no arbitrage is using the martingale measure.

Definition 2.5. (Martingale) *We say Q is a martingale measure for $\mathcal{A}$ if*

(i) Q is a probability measure on $(\Omega, \mathcal{F})$ such that, for any, $\omega \in \Omega$, $Q(\omega) > 0$, and

(ii) for any t,

$$\mathbf{E}^Q(S_t \mid \mathcal{F}_{t-1}) = S_{t-1}.$$

We will use $\mathcal{M}(\mathcal{A})$ to denote the set of all martingale measures for $\mathcal{A}$.

In other words, under a martingale measure the expected discounted price of any risky assets always remains the same. Alternatively, one can interpret a martingale measure as a way to represent a view that risky and risk-less assets are treated with no difference. Thus, a martingale measure is also called a risk neutral measure.

The following fundamental theorem of asset pricing relates the no arbitrage principle and the existence of a martingale measure. The proof relies on the convex separation theorem (see e.g. [5, Section 4.3]).

Theorem 2.6. (No Arbitrage and Martingale) *Let $\mathcal{A}$ be a financial market model with finite period T. Then the following are equivalent*

(i) there are no arbitrage trading strategies;

(ii) $\mathcal{M}(\mathcal{A}) \neq \emptyset$.

Proof. The implication (ii)→(i) follows directly from the definition. Let $Q \in \mathcal{M}(\mathcal{A})$. If $\Theta \in \mathcal{T}(\mathcal{A})$ is an arbitrage trading strategy, then, for some t, $G_t(\Theta) > 0$ and consequently $E^Q(G_t(\Theta)) > 0$, a contradiction.

To prove (i)→(ii) observe that, $G_T(\mathcal{T}(\mathcal{A})) \cap \mathrm{int} RV(\Omega)_+ = \emptyset$. Since $G_T(\mathcal{T}(\mathcal{A}))$ is a subspace, by the convex set separation theorem $G_T(\mathcal{T}(\mathcal{A}))^\perp$

contains a vector q with all components positive. We can scale q to a probability measure Q. Then it is easy to check $Q \in \mathcal{M}(\mathcal{A})$. Q.E.D.

In practice, investors always face scenarios in which not all the trading strategies in $\mathcal{T}(\mathcal{A})$ are available. For example, if in any trading margin is not allowed, then the set of admissible trading strategies is defined by

$$\mathcal{T}_+(\mathcal{A}) = \{\Theta \in \mathcal{T}(\mathcal{A}) \mid \Theta_0^t \geq 0, t = 1, \ldots, T\}.$$

By choosing different subset $\mathcal{T} \subset \mathcal{T}(\mathcal{A})$ we can conveniently handle different scenarios of the financial model with assets $\mathcal{A}$ over economy $(\Omega, \mathcal{F}, P)$. We can view various questions related to these scenarios as to find suitable admissible trading strategies to arrive at preferred risk adjusted gains. How to model the preference then?

3 Preference

By and large, there are two ways of modeling the preference of market agents: using concave utility functions and using convex risk or loss functions. As a result, problems related to these financial models will be handled in the framework of maximizing expected utility functions or minimizing convex risk functions both rely heavily on convex analysis.

3.1 Utility function

Experience in our daily life tells us that mathematical expectation is often not what people use to compare payoffs with uncertainty. Two common examples are lottery and insurance. Both come with price that are higher than the expected payoffs because lottery authorities and insurance companies make a lot of money. Yet people buy them anyway. Economists explain this using utility functions: people are usually comparing the expected utility. They hypothesize that the utility function is increasing reflecting the more the better and the marginal utility decreases as the quantity increases. The latter is also interpreted as the tendency of risk aversion: the more we have the less we are willing to risk. Thus, people are willing to take a large risk of losing the relatively small amount of money of buying a lottery ticket or paying the insurance premium in exchange for the unlikely events of winning the lottery or get insurance compensation. This idea goes back at least to Daniel Bernouli who used the log utility function $u(x) = \ln(x)$ in his solution to the

St. Petersburg wager problem. Now we usually model utility function using an increasing concave function. There are many increasing concave functions. Besides the log utility, power utility functions $(x^{1-\gamma}-1)/(1-\gamma), \gamma > 0$ and the exponential utility functions $-e^{-\alpha x}, \alpha > 0$ are also frequently used. We note that $\ln(x) = \lim_{\gamma \to 0}(x^{1-\gamma}-1)/(1-\gamma)$.

In dealing with a particular application problem the choice of the utility function is often base on economic or tractability considerations. Different people can have different utility functions that reflects their own attitude towards the trade-off between rewards and risks.

For our mathematical model, it is important to know what kind of general conditions we should impose on a utility function. We consider a general extended valued upper semicontinuous utility function u. The following is a collection of conditions that are often imposed in financial models:

(u1) (Risk aversion) u is strictly concave,

(u2) (Profit seeking) u is strictly increasing and $\lim_{t \to +\infty} u(t) = +\infty$,

(u3) (Bankruptcy forbidden) For any $t < 0$, $u(t) = -\infty$ and $\lim_{t \to 0+} u(t) = -\infty$,

(u4) (Standardized) $u(1) = 0$ and u is differentiable at $t = 1$.

3.2 Risk measure

An alternative to maximizing utility functions is to minimize risks. There are many different ways of measuring risks. One of the pioneering work in this area was Markowitz's portfolio theory in which Markowitz measures the risks using the variation. This results in a quadratic risk function. Since Markowitz's work many different risk measures have been proposed. Let us consider a T period economy $(\Omega, \mathcal{F}, P)$ and let $\mathcal{T}$ be the set of self-financing trading strategies on this economy. Starting with an initial wealth w_0 and using a trading strategy $\Theta \in \mathcal{T}$, the final payoff at time T is $X = G_T(\Theta) + w_0 \in RV(\Omega)$. For simplicity we focus on the final payoff. Then one way to model the risks is to use a risk measure $\rho : RV(\Omega) \to R$. For example, in Markowitz's portfolio theory, the problems is to

$$\begin{aligned} \text{minimize} \quad & \rho(X) \\ \text{subject to} \quad & \mathbf{E}(X) = \mu, X \in \mathcal{X}. \end{aligned} \tag{3.1}$$

Here $\rho(X) = Var(X)$ and $\mathcal{X} = \{G_T(\Theta) + w_0 : \Theta \in \mathcal{T}\}$ signifies the set of payoffs of all allowable portfolios. Sometimes, we may wish to ensure the risk is below a certain level. Then instead of considering optimization problem we consider a viability problem of $\rho(X) \leq L$, where L is a constant representing the risk that a particular agent is willing to tolerant. We now turn to desirable properties of risk measures.

(r1) (Convexity) for $X_1, X_2 \in RV(\Omega)$ and $\lambda \in [0, 1]$,

$$\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda\rho(X_1) + (1 - \lambda)\rho(X_2),$$

(r2) (Monotone) $X_1 - X_2 \in RV(\Omega)_+$ implies $\rho(X_1) \leq \rho(X_2)$.

Convexity means diversification will reduce the risk. Monotonicity is also intuitive, it says that a dominate random variable has a smaller risk. One may measure the risk of $X \in RV(\Omega)$ by the minimum amount of additional capital reserve to ensure that there is no risk of bankruptcy. If ρ is such a risk measure then the following properties are natural:

(r3) (Translation property) $\rho(Y + c\vec{1}) = \rho(Y) - c$ for any $Y \in RV(\Omega)$ and $c \in R$,

(r4) (Standardized) $\rho(0) = 0$.

These properties are gradually emerged in analyzing the risks. Modeling the risk control of market makers of exchanges, Artzner, Delbaen, Eber and Heath [2] introduced the influential concept of coherent risk measure: ρ is a positive homogeneous and subadditive function. This is later generalized to convex risk measure by Föllmer and Schied [15, 16], Frittelli and Gianin [17] and Rudloff [38]. An important concrete convex risk measure is the conditional value at risk proposed by Rockafellar and Uryasev [35, 36] that generalizes the widely used concept of value at risk (see [22]). As we can see, involving convex risk measure in the model also leads to the use of convex analysis.

4 Portfolio theory

Portfolio theory considers a one period financial model. In this case a trading strategy $\Theta \in \mathcal{T}(\mathcal{A})$ is simply a constant vector $\Theta = (\theta_0, \theta_1, \ldots, \theta_M)$ where each θ_m represents the weight of asset a_m in the portfolio. The question is what is the best portfolio. Since different agents have different preferences there is no unique answer to this question.

4.1 Markowitz portfolio

Markowitz considered only risky assets in his pioneering portfolio theory [30]. The idea is that for a fixed expected return one should choose portfolios with minimum variation, which serves as a measure for the risk. In general, a portfolio with a higher expected return also accompanied with a higher variation (risk). The tradeoff is left to the individual agent.

Use $\hat{S} = (S^1, \ldots, S^M)$ to denote the price process of the risky assets and $\hat{\Theta} = (\theta_1, \ldots, \theta_M)$ to denote the portfolio. For a given expected payoff r_0 and an initial wealth w_0 we can formulate the problem as

$$\begin{aligned} \text{minimize} \quad & \operatorname{Var}(\hat{\Theta} \cdot \hat{S}_1) \\ \text{subject to} \quad & \mathbf{E}[\hat{\Theta} \cdot \hat{S}_1] = r_0 \\ & \hat{\Theta} \cdot \hat{S}_0 = w_0. \end{aligned} \tag{4.1}$$

Regarding $\hat{S}$ as a row vector of random variables and $\hat{\Theta}$ as a row vector, denoting $\mathbf{E}(\hat{S}_1) = [\mathbf{E}(\hat{S}_1^1, \ldots, \mathbf{E}(\hat{S}_1^M)]$,

$$A = \begin{bmatrix} \mathbf{E}(\hat{S}_1) \\ \hat{S}_0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} r_0 \\ w_0 \end{bmatrix},$$

we can rewrite (4.1) as an entropy maximization problem

$$\begin{aligned} \text{minimize} \quad & f(x) := \frac{1}{2} x^\top \Sigma x \\ \text{subject to} \quad & Ax = b. \end{aligned} \tag{4.2}$$

Here $x = \hat{\Theta}^\top$ and

$$\Sigma = (\mathbf{E}[(S_1^i - \mathbf{E}(S_1^i))(S_1^j - \mathbf{E}(S_1^j))])_{i,j=1,\ldots,M}. \tag{4.3}$$

The coefficient $1/2$ is added to the risk function to make the computation easier. Clearly, Σ is a symmetric positive semidefinite matrix. We will assume that it is in fact positive definite. We will use the Fenchel duality theory to analyze this problem. Recall that, for a convex function $\phi : X \to R \cup \{+\infty\}$, its dual is defined by

$$\phi^*(y) = \sup_{x \in X} \{\langle y, x \rangle - \phi(x)\} : X^* \to R \cup \{+\infty\}.$$

It is easy to calculate that

$$f^*(y) = \frac{1}{2} y^\top \Sigma^{-1} y. \tag{4.4}$$

It follows from the Fenchel duality Theorem (see e.g. [5, Section 4.4.]) that the value of problem (4.2) equals to that of its dual:

$$\begin{aligned} \text{maximize} \qquad & b^\top y - \frac{1}{2} y^\top A \Sigma^{-1} A^\top y \\ & = \frac{1}{2} b^\top (A \Sigma^{-1} A^\top)^{-1} b. \end{aligned} \tag{4.5}$$

Denote σ the minimum standard deviation of portfolios with expected return r_0, we have

$$\sigma^2 = b^\top (A \Sigma^{-1} A^\top)^{-1} b. \tag{4.6}$$

Let $\bar{x}$ be the solution of (4.2). We have

$$\begin{aligned} \frac{1}{2} \bar{x}^\top \Sigma \bar{x} &= \frac{1}{2} b^\top (A \Sigma^{-1} A^\top)^{-1} b \qquad (4.7) \\ &= \frac{1}{2} b^\top (A \Sigma^{-1} A^\top)^{-1} A \Sigma^{-1} \Sigma \Sigma^{-1} A^\top (A \Sigma^{-1} A^\top)^{-1} b. \end{aligned}$$

Since the solution to problem (4.2) is unique, it follows that

$$\bar{x} = \Sigma^{-1} A^\top (A \Sigma^{-1} A^\top)^{-1} b. \tag{4.8}$$

Define $\alpha = \mathbf{E}(\hat{S}_1) \Sigma^{-1} \mathbf{E}(\hat{S}_1)^\top$, $\beta = \mathbf{E}(\hat{S}_1) \Sigma^{-1} \hat{S}_0^\top$ and $\gamma = \hat{S}_0 \Sigma^{-1} \hat{S}_0^\top$. Assuming that $\mathbf{E}(\hat{S}_1)$ is not proportional to S_0, then the Cauchy-Schwarts inequality implies that

$$\alpha\gamma - \beta^2 > 0. \tag{4.9}$$

We can summarize the above conclusion about the minimum risk and the corresponding portfolio as:

Theorem 4.1. *For given initial wealth w_0 and expected payoff r_0, the minimum risk σ and the corresponding minimum risk portfolio Θ are determined by*

$$\sigma(r_0, w_0) = \sqrt{\frac{\gamma r_0^2 - 2\beta r_0 w_0 + \alpha w_0^2}{\alpha\gamma - \beta^2}} \tag{4.10}$$

and

$$\Theta(r_0, w_0) = \frac{\mathbf{E}(\hat{S}_1)(\gamma r_0 - \beta w_0) + \hat{S}_0(\alpha w_0 - \beta r_0)}{\alpha\gamma - \beta^2}\Sigma^{-1} \tag{4.11}$$

Note that both $\sigma(r_0, w_0)$ and $\Theta(r_0, w_0)$ are homogeneous functions we have

Corollary 4.2. *Use μ to denote the expected return on unit initial wealth and let $\sigma = \sigma(\mu, 1)$ and $\Theta = \Theta(\mu, 1)$. Then*

$$\sigma = \sqrt{\frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2}} \tag{4.12}$$

and

$$\Theta = \frac{\mathbf{E}(\hat{S}_1)(\gamma\mu - \beta) + \hat{S}_0(\alpha - \beta\mu)}{\alpha\gamma - \beta^2}\Sigma^{-1} \tag{4.13}$$

Moreover, $\sigma(\mu w_0, w_0) = w_0\sigma$ and $\Theta(\mu w_0, w_0) = w_0\Theta$.

If our sole goal is to minimize the risk then our problem becomes

$$\begin{aligned} \text{minimize} \qquad & f(x) := \frac{1}{2}x^\top \Sigma x \\ \text{subject to} \qquad & \hat{S}_0^\top x = w_0. \end{aligned} \tag{4.14}$$

Using a similar argument one can show

Theorem 4.3. *The minimum risk portfolio is*

$$\Theta_{\min} = \gamma^{-1} w_0 \hat{S}_0 \Sigma^{-1}$$

and its standard deviation is

$$\sigma_{\min} = \gamma^{-1/2} w_0.$$

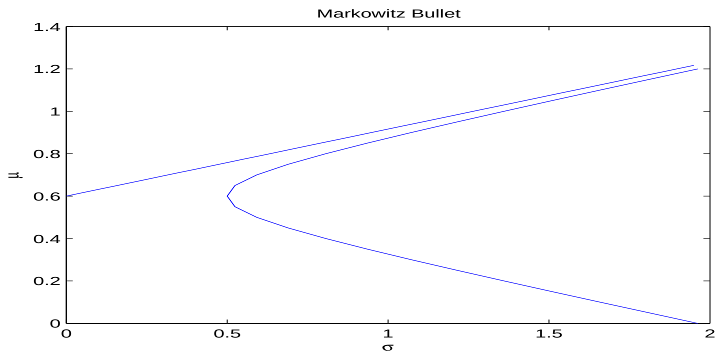

Figure 1: Markowitz Bullet and its asymptote

We now turn to a geometric interpretation of the Markowitz portfolio theory. Note that (4.12) determines μ as a function of σ. Drawing this function on the $\sigma\mu$-plan we get the curve in Figure 1, which is commonly known as a Markowitz bullet for its shape.

Every point inside the Markowitz bullet represents a portfolio that can be moved horizontally to the left to a point on the boundary of the bullet. This point on the boundary represents a portfolio with the same expected return but less risk. For every point on the lower half of the boundary of the Markowitz bullet, one can find a corresponding point on the upper half of the boundary with the same variation and a higher expected return. Thus, preferred portfolios are represented by points on the upper boundary of the Markowitz bullet. Along this upper boundary of the Markowitz Bullet we can trade-off between risk and return: moving to the upper-left along the curve increases both the return and the risk.

We note that the upper boundary of the boundary has an asymptote whose slope can be determined by

$$\lim_{\sigma\to\infty}\frac{\mu}{\sigma}=\sqrt{\frac{\alpha\gamma-\beta^2}{\gamma}}. \tag{4.15}$$

By taking the limit of the tangent line of points on the boundary of the Markowitz bullet one can show that this asymptote is

$$\mu=\frac{\beta}{\gamma}+\sqrt{\frac{\alpha\gamma-\beta^2}{\gamma}}\sigma. \tag{4.16}$$

The μ intercept of this asymptote is β/γ. This number will play an important role in our discussion of the capital asset pricing model.

4.2 Capital asset pricing model

Capital asset pricing model (CAPM) is a generalization of the Markovwitz portfolio theory by allowing risk free asset in the portfolio. It was introduced independently by Sharpe [41], John Lintner [28] and Mossin [32]. Similar to (4.1) we now face the problem of

$$\begin{array}{rl} \text{minimize} & \mathrm{Var}(\Theta \cdot S_1) \\ \text{subject to} & \mathbf{E}[\Theta \cdot S_1] = \mu \\ & \Theta \cdot S_0 = 1. \end{array} \tag{4.17}$$

Here we standardized the initial wealth to 1 and μ is the expected return. Since $\mathrm{Var}(S_1^0) = 0$ one can show that

$$\mathrm{Var}(\Theta \cdot S_1) = \mathrm{Var}(\hat{\Theta} \cdot \hat{S}_1). \tag{4.18}$$

Relation (4.18) suggests a strategy of solving problem (4.17) in two steps. First, for a portfolio with $\theta = \theta_0 \geq 0$, denote $R = S_1^0/S_0^0$, the return on the risk free asset, we solve problem

$$\begin{array}{rl} \text{minimize} & \mathrm{Var}(\hat{\Theta} \cdot \hat{S}_1) \\ \text{subject to} & \mathbf{E}[\hat{\Theta} \cdot \hat{S}_1] = \mu - \theta R \\ & \hat{\Theta} \cdot \hat{S}_0 = 1 - \theta. \end{array} \tag{4.19}$$

Then, we minimize the minimum variation of (4.19) as a function of θ.

By Theorem 4.1 the minimum variation corresponding to problem (4.19) as a function of θ is determined by

$$\begin{aligned} f(\theta) &= [\sigma(\mu - \theta R, 1 - \theta)]^2 \\ &= \frac{\gamma(\mu - \theta R)^2 - 2\beta(\mu - \theta R)(1 - \theta) + \alpha(1 - \theta)^2}{\alpha\gamma - \beta^2} \end{aligned} \tag{4.20}$$

Since the coefficient of θ^2 in the quadratic function $f(\theta)$ is $(\gamma R^2 + \alpha)/(\alpha\gamma - \beta^2) > 0$, the solution of problem (4.17) corresponds to the minimum of function f that attains at

$$\bar{\theta} = \frac{\alpha - \beta R + (\gamma R - \beta)\mu}{\alpha - 2\beta R + \gamma R^2}, \tag{4.21}$$

the solution to the equation $f'(\theta) = 0$. We note that by inequality (4.9) the quadratic form of R in the denominator is always positive. It is easy to see that the share invested in the risky assets is

$$1 - \bar{\theta} = \frac{\beta - \gamma R}{\alpha - 2\beta R + \gamma R^2}(\mu - R) \tag{4.22}$$

Note that the risky asset is involved in the minimum variance portfolio only when $1 - \bar{\theta} > 0$ or

$$R < \beta/\gamma \tag{4.23}$$

by (4.22). Let us first focus on the case when R satisfies (4.23). In this case, $\bar{\theta}$, the share of the risk-less asset in the portfolio is determined by (4.21). Including risky assets we always expect to get a higher return than the risk free assets, we are only interested in $\mu \geq R$. It is easy to check that when $\mu = R$, $\bar{\theta} = 1$. Intuitively, this is to say if our goal is to achieve a return of R, then we should choose a portfolio that contains only the riskless asset which has the minimum variance $\sigma = 0$. Another interesting μ value is $\mu_M = \frac{\alpha - \beta R}{\beta - \gamma R}$ corresponding to $\bar{\theta} = 0$. This tells us that in order to achieve the return μ_M, we must put all the money in the risky assets. What is the appropriate portfolio? Let us set $\mu = \mu_M$ and $\theta = 0$ in (4.19) which leads to the following portfolio problem:

$$\begin{aligned} \text{minimize} \quad & \mathrm{Var}(\hat{\Theta} \cdot \hat{S}_1) \\ \text{subject to} \quad & \mathbf{E}[\hat{\Theta} \cdot \hat{S}_1] = \mu_M = \frac{\alpha - \beta R}{\beta - \gamma R} \\ & \hat{\Theta} \cdot \hat{S}_0 = 1. \end{aligned} \tag{4.24}$$

By Theorem 4.1, we derive the optimal portfolio of (4.24) to be

$$\Theta_M = \frac{\mathbf{E}(\hat{S}_1) - R\hat{S}_0}{\beta - \gamma R}\Sigma^{-1} = \frac{\mathbf{E}(\hat{S}_1) - R\hat{S}_0}{(\mathbf{E}(\hat{S}_1) - R\hat{S}_0)\Sigma^{-1}\hat{S}_0^{\top}}\Sigma^{-1}. \tag{4.25}$$

This is often referred to as the *capital market portfolio*. Also by Theorem 4.1, the minimum variance corresponding to the capital market portfolio is

$$\sigma_M = \frac{\sqrt{\gamma(\alpha - \beta R)^2 - 2\beta(\alpha - \beta R)(\beta - \gamma R) + \alpha(\beta - \gamma R)^2}}{\sqrt{\alpha\gamma - \beta^2}(\beta - \gamma R)} \tag{4.26}$$

Clearly, the point (σ_M, μ_M) lies on the boundary of the Markowitz bullet.

In general, for $\mu > R$ and $\mu \neq \mu_M$, setting $\theta = \bar{\theta}$ in (4.19) we have the portfolio problem

$$\begin{aligned} \text{minimize} \quad & \mathrm{Var}(\hat{\Theta} \cdot \hat{S}_1) \\ \text{subject to} \quad & \mathbf{E}[\hat{\Theta} \cdot \hat{S}_1] = \mu_M \frac{\mu - R}{\mu_M - R} \\ & \hat{\Theta} \cdot \hat{S}_0 = \frac{\mu - R}{\mu_M - R}. \end{aligned} \tag{4.27}$$

By Corollary 4.2, the minimum variance of problem (4.27) and its corresponding risky portfolio are

$$\sigma = \frac{\mu - R}{\mu_M - R} \sigma_M \tag{4.28}$$

and

$$\Theta = \frac{\mu - R}{\mu_M - R} \Theta_M \tag{4.29}$$

In view of (4.18), formula (4.28) also determines the minimum variance of problem (4.17). Thus, we have the following linear relationship between the expected return μ and the minimum variance σ in problem (4.17).

$$\mu = \frac{\mu_M - R}{\sigma_M} \sigma + R. \tag{4.30}$$

Geometrically, this is a line on the $\sigma\mu$-plane called the *capital market line*. For any expected return $\mu > R$ and the corresponding minimum variation σ solving problem (4.17), the point (σ, μ) lies on the capital market line. Moreover, the pair (σ, μ) derived from solving problem (4.17) is always better than restricting problem (4.17) to allowing only risky assets. It follows that the capital market line is always above the Markowitz bullet and tangent to the latter at (σ_M, μ_M) as shown in Figure 2. Thus, the capital asset pricing model is a true generalization of the Markowitz portfolio theory.

What happens when $R > \beta/\gamma$? We can see from (4.22) that in this case $1 - \bar{\theta} < 0$ indicting short selling the risky assets. If we allow short selling then the analysis is similar. The only difference is that now the capital market line lies below the Markowitz bullet as shown in Figure 3. Note that the

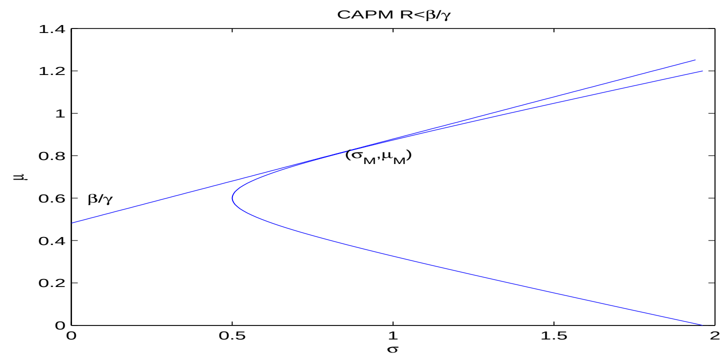

Figure 2: Capital market line $R < \beta/\gamma$

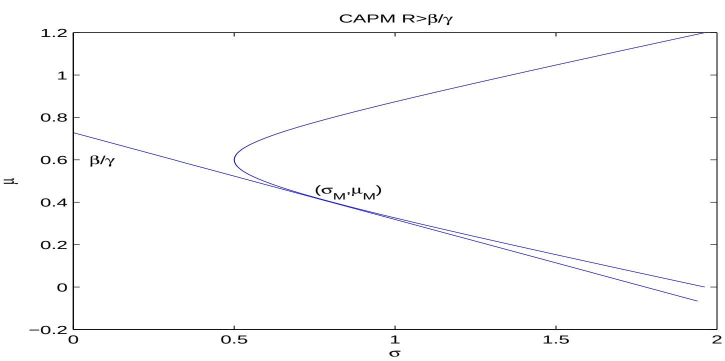

Figure 3: Capital market line $R > \beta/\gamma$

useful capital allocation lies on the part of the capital market line to the left of the μ axis (not shown in Figure 3) where the negative σ value should be read as $|\sigma|$. The practical implication is that when the bond yield is greater than the critical value β/γ one should definitely not invest in stocks, if not short selling them.

4.3 Sharpe ratio

Thinking a little we will realize that to construct the capital market portfolio, theoretically, we need to use every available risky asset. Given the huge number of available equities constructing the capital market portfolio is practically impossible even if we have accurate probability distribution information on all the available risky assets (which is another impossible task). Thus, we have to deal with less than optimal situation. What happens if we mix risk free asset with an arbitrary portfolio of risky assets (not necessarily the capital market portfolio)? Let $\hat{\Theta} = (\theta_1, \ldots, \theta_M)$ be such a portfolio corresponding to risky assets $(a^1, \ldots, a^M)$ with price random vector $\hat{S} = (S^1, \ldots, S^M)$. Again we standardize the portfolio so that $\hat{\Theta} \cdot \hat{S}_0 = 1$. Denote $\mu^* = \mathbf{E}(\hat{\Theta} \cdot \hat{S}_1)$ and $\sigma^* = \sqrt{\mathrm{Var}(\hat{\Theta} \cdot \hat{S}_1)}$. Then any mix of this portfolio with a risk free asset having return R will produce a portfolio whose expected return μ and standard deviation σ lies on the line

$$\mu = \frac{\mu^* - R}{\sigma^*}\sigma + R. \tag{4.31}$$

Portfolios of risky assets with larger $\frac{\mu^*-R}{\sigma^*}$ have the potential of generating mix with risk free asset that have smaller variation for a fixed expected return. Sharpe propose to compare risky portfolios such as those maintained by mutual funds using this idea [42]. As an illustration, suppose that $R_1, \ldots, R_N$ are the monthly return of a mutual fund a in the past N months and the monthly return of the risk free asset is R. Define a random variable X with finite values $\{R_n - R \mid n = 1, \ldots, N\}$ and $\mathrm{prob}(X = R_n - R) = 1/N$. Then the Sharpe ratio of a is defined as

$$s(a) = \frac{\mathbf{E}(X)}{\sqrt{Var(X)}}. \tag{4.32}$$

We can see that the Sharpe ratio is in fact a statistical estimate of $\frac{\mu^*-R}{\sigma^*}$. Now Sharpe ratio has become one of the most popular standard for comparing the performances of mutual funds and other investment methods.

Markowitz portfolio theory, the Capital asset pricing model and the Sharpe ratio are all based on using variation as a risk measure. It is an interesting question to ask what happens if we replace the variation with a different convex risk measure?

4.4 Capital growth portfolio

The capital growth portfolio theory assumes that the portfolio manager maximizes the expected log utility. If we think the distribution of the risky assets are empirically derived from repeated sampling of the performance of these assets in consecutive investment intervals, then maximizing the expected log utility amounts to maximizing expected compounded return of the portfolio and thus the name of growth portfolio. Such a portfolio is sometimes referred to as the Kelly portfolio since Kelly is the first to suggest maximizing the log utility to derive the best betting size in gambling problem as a way to understand Shannon information rate [25, 40]. In this connection the capital growth portfolio can also be viewed as the one that best utilizes all the information implied in the price random vector. Thorp [43], Thorp and Kassouf[45] generalized Kelly's method to deal with gambling and investment problems. Optimal log utility for the growth portfolio can be used as an indicator of the effectiveness of different investment methods as discussed in [50]. The risk free asset is part of the portfolio so that one don't have to put all the capital in risky assets. Let the initial wealth be w_0 then the optimization problem is for $\Theta \in R^M$,

$$\begin{aligned} \text{maximize} \quad & \mathbf{E}[\ln(w_0 + G_1(\Theta))] && (4.33) \\ \text{subject to} \quad & \Theta = (\theta_0, \theta_1, \ldots, \theta_M) \in R^{M+1}. && (4.34) \end{aligned}$$

To understand the behavior of a growth portfolio let us look at the simplest case when there is only one risky asset. In this case the problem is equivalent to maximizing $\ln(1 + sx)$ where $s = \theta_1$ is the size of the position on the risky asset and x is the percentage return to the risky asset. Assuming $\mathbf{E}x > 0$ we consider $s \in [0, \infty)$ that means (unlimited) margin is allowed. A typical return function is given in Figure 2. We can see that as the size of the risky position increases so is the return in the beginning. However, after the return reach the maximum, the return begins to inversely related to the size of the risky asset. The message is that if the expected discounted return is positive one should invest in risky asset but not over leverage. While the growth portfolio provide us with useful insight, one of the shortcomings of this model is that the optimal solution is usually too sensitive to perturbations. Since no financial model is perfect, practitioners seldom attempt to use the growth portfolio. Often a scale back from the estimated optimal risky positions is used to tame the risk. Vince's fractional Kelly criterion [47, 48] is one example which developed earlier work of Thorp and Kassouf [45, 44].

5 Pricing financial derivatives

Financial derivatives are financial instruments whose values depend on other financial assets. The suspects related to the 2008 financial crisis such as

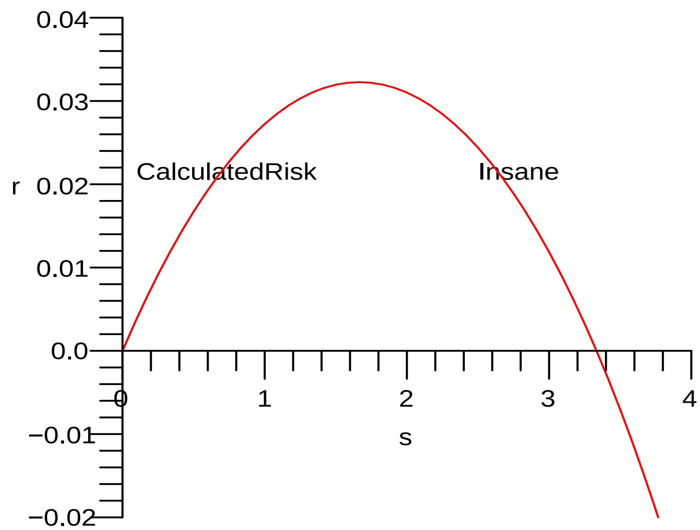

Figure 4: A typical log return function

Collateral Debt Obligations, Mortgage Backed Securities and Special Purpose Vehicles are all financial derivatives (see [20]). How to pricing these financial derivatives is one of the central issues in modern financial theory. We will illustrate the role of convex analysis in this area using the problem of pricing options – one of the simplest financial derivatives.

5.1 Options and a one period model example

Option is a contract that grants the holder a right to buy or sell a certain asset at a prescribed price. Take a stock index for example. There are two kinds of basic options: calls and puts. For simplicity we consider only European style options which clears only at maturity. A call (put) option of a stock index with a strike price K gives the holder of the option the right to buy (sell) the underlying stock index at price K at its maturity. If the value of the stock index is S at maturity then the call option worth $(S-K)^+ := \max(S-K, 0)$ and the put option worth $(S-K)^- := \max(K-S, 0)$. The pricing problem with regards to the option is what should be its price now.

We will use an (overly) simplified one period model to explain the idea of prevailing option pricing theory. Consider a stock index whose current price (called the spot price) is standardized to $S_0 = 1$ and an call option with a strike price 1 (at the money). We assume a one period model in which the stock index and its options can only be traded now when $t = 0$ and at the maturity (M) of the option represented by $t = 1$. Assumes that S_1 can take only two values 0.5 or 2. Then the value of the stock index and its option at maturity is represented in Diagram (5.1):

$$
\begin{array}{ccccccc}
\text{Stock} & & \text{Maturity} & \qquad & \text{Option} & & \text{Maturity}\\
 & \overset{0.5}{\nearrow} & 2 & & & \overset{0.5}{\nearrow} & 1\\
1 & & & & c & & \\
 & \underset{0.5}{\searrow} & 0.5 & & & \underset{0.5}{\searrow} & 0
\end{array} \tag{5.1}
$$

Using c to represent the current price of the option. It is tempting to use the expected value of the option, which is 0.5 in this case, but Black and Scholes [3] suggested an ingenious method that in this simple case comes down to assemble a portfolio by buying $2/3$ shares of the stock index and borrowing $1/3$ cash (assuming 0 interest rate or equivalently the price of the stock index has discounted the interest rate) then the cost is $p = 2/3 - 1/3 = 1/3$. We can check that at maturity the payoff of this portfolio is exactly the same as that of the option. Thus, we should have $c = p = 1/3$. If the price of the option deviated from this value an arbitrage opportunity occurs.

The right mix for our portfolio is, of course, coming from solving the vector equation

$$stock * \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} + cash * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.2}$$

This method is referred to as replicating portfolio option pricing. A draw back of this method is that if one changes the option strike price K then one needs to solve a different equation (5.2) again. Cox and Ross [6] proposed a different approach. They first consider a risk neutral world in which market agents has no risk a version and therefore all the assets should have the same rate of return regardless of their risks. Base on this one can deduce a probability measure for each possible scenario called a risk neutral measure. Then the price of the option can be determined using this risk neutral measure. Finally, they argue that when the option can be replicated the price derived in the risk neutral world should be the same as that in the real world because the replicating portfolio will eliminate all the risks. Again we will use the above one period model to illustrate this method. Denote the risk neutral measure by $\pi = (\pi_1, \pi_2)$ then since this is a probability measure we have

$$\pi_1 + \pi_2 = 1. \tag{5.3}$$

Since in a risk neutral world the stock index and the cash should earn the same expected return we have

$$\pi_1 2 + \pi_2 0.5 = 1. \tag{5.4}$$

Solving (5.3) and (5.4) we get $\pi_1 = 1/3$ and $\pi_2 = 2/3$. Now we can using the risk neutral measure to calculate the option price

$$c = \pi_1 \cdot 1 + \pi_2 \cdot 0 = \frac{1}{3}.$$

Note that the option price derived this way is the same as the one derived from the replicating portfolio. However, once we calculated the risk neutral probability we can use it to calculate the price of options with a different strike price handily. This convenience made the risk neutral pricing method rather popular in financial derivative pricing.

Besides the ease of computation, one of the most important feature that makes the portfolio replicating and risk neutral option pricing popular is the general believe that they provide pricing of the option independent of individual market agent's preference. This uniform pricing made the trading of options and other derivatives easier and largely accounted for the explosive increase in the trading volume of the financial derivatives before the 2008 financial crisis. However, the claim that these pricing mechanisms are independent of the preference of market agents is inaccurate. In fact, we show in what follows that the replicating portfolio pricing can be derived by maximizing a portfolio of stock index and cash with respect to a particular class of utility functions and the risk neutral probability is a natural consequence of solving the dual problem. Thus, it appears that they are merely a special case of the traditional investment strategy of maximizing utility that accounts for the risk aversion of the market participants.

5.2 General Model

Given a set of assets $\mathcal{A}$. A contingent claim (financial derivative) is a random variable whose payoff is related to that of the assets in $\mathcal{A}$. Here we assume that the contingent claim can only be traded at $t = 0$ and $t = T$. Let $H \in RV(\Omega)$ be the payoff random variable for such a contingent claim, and let H_0 be its price at $t = 0$. Including H in the set of assets. Then a self-financing trading strategy for $\mathcal{A} \cup \{H\}$ has the form (Θ, β) where $\Theta \in \mathcal{T}(\mathcal{A})$ and $\beta \in R$ is a real number representing quantity of the contingent claims in the trading strategy. Between 0 and T, β is a constant because the contingent claim cannot be traded. We use utility function u to represent the risk aversion of the agent and assume u satisfies conditions (u1)-(u4). Again, assume a standardized initial wealth $w_0 = 1$, we face the optimization problem

$$\begin{aligned} \max \quad & (E(u(y))) \\ \text{subject to} \quad & y \in G_T(\mathcal{T}(\mathcal{A})) + \beta(H - H_0) + 1. \end{aligned} \tag{5.5}$$

Define $f(y) = -E(u(y))$ and $g(y) = \iota_{G_T(\mathcal{T}(\mathcal{A}))+R(H-H_0)+1}(y)$, where ι_A is the indicator function of set A defined by $\iota_A(x) = 0$ for $x \in A$ and $\iota_A(x) = +\infty$ for $x \notin A$. Then we can rewrite problem (5.5) as

$$-\min_y \{f(y) + g(y)\} \tag{5.6}$$

The dual problem is,

$$
\begin{aligned}
& -\max\{-f^*(z) - g^*(-z)\} && (5.7)\\
= \ & \min\ \sigma_{1+R(H-H_0)+G_T(\mathcal{T}(\mathcal{A}))}(-z) + \mathbf{E}((-u)^*(z)) \\
= \ & \min\ \{\langle 1+\beta(H-H_0), -z\rangle + \iota_{G_T(\mathcal{T}(\mathcal{A}))^\perp}(z) \mid z > 0, \beta \in R\} \\
= \ & \min\ \{-\langle 1, z\rangle \mid z > 0,\ z \in G_T(\mathcal{T}(\mathcal{A}))^\perp,\ \langle z, H - H_0\rangle = 0\},
\end{aligned}
$$

where $\sigma_A = \iota_A^*$ is the support function of set A. We assume that the utility function satisfies condition (u1)-(u4). Again, we can check that the CQ condition

$$1 \in \mathrm{cont} f \cap \mathrm{dom} g = RV(\Omega) \cap (G_T(\mathcal{T}(\mathcal{A})) + R(H - H_0) + 1)$$

holds so that there is no arbitrage if and only if both primal and dual problem have the same finite solution. Thus, we have

Theorem 5.1. *There is no arbitrage trading strategy for a financial market $\mathcal{A} \cup \{H\}$ if and only if*

$$H_0 \in \{\mathbf{E}^Q(H) \mid Q \in \mathcal{M}(\mathcal{A})\}.$$

Proof. We already know that there is no arbitrage trading strategy for assets $\mathcal{A} \cup \{H\}$ if and only if the value of the optimization problem (5.5) is finite. Since $1 \in \mathrm{dom}\ g \cap \mathrm{int}(\mathrm{dom}\ f)$, we see that the CQ condition is satisfied for the Fenchel duality theorem. Thus, when the prime problem (5.6) is finite so is the dual problem (5.7). The dual problem (5.7) has a finite value if and only if, there exists $z > 0$

$$z \in G_T(\mathcal{T}(\mathcal{A}))^\perp,\ \langle z, H - H_0\rangle = 0. \tag{5.8}$$

Define $Q = zP/\int_\Omega z dP$. Then $Q \in \mathcal{T}(\mathcal{A})$ and $\mathbf{E}^Q(H - H_0) = 0$. Thus, there is no arbitrage if and only if

$$H_0 \in \{\mathbf{E}^Q(H) \mid Q \in \mathcal{M}(\mathcal{A})\}. \tag{5.9}$$

Q.E.D.

In general there exists many martingale measures and thus the inclusion (5.9) only gives us a range of the price H_0 for the contingent claim. Additional assumptions are needed to determine a price. Using entropy maximization gained attention recently (see [4, 46]). Using this method one selects among martingale measures the one that maximizing an entropy. Heuristically, such a selection is relatively smooth so that (approximate) arbitrage becomes difficult. When using the (relative) Shannon entropy it means to select the martingale measure with the most information (under given prior empirical knowledge). In the context of the duality relationship between the utility maximization and risk neutral measure pricing, selecting a risk neutral measure that maximizing a particular (relative) entropy amounts to select a particular utility function. This could provide additional validation on the appropriateness the entropy maximization approach. We note that to recover of the utility that corresponding to a particular choice of the (relative) entropy is often a challenge.

5.3 Complete markets and arbitrage

Observe that, if $\mathcal{M}(\mathcal{A})$ is singleton, then the price H_0 of the contingent claim is unique. The following theorem characterizes when this happens.

Theorem 5.2. *Suppose, $\mathcal{M}(\mathcal{A}) \neq \emptyset$. Then, $\mathcal{M}(\mathcal{A})$ is singleton, if and only if $R := \{\Theta_T \cdot S_T \mid \Theta \in \mathcal{T}(\mathcal{A})\} = RV(\Omega)$.*

Proof. If part: Let, $Q_1, Q_2 \in \mathcal{T}(\mathcal{A})$. Then,

$$\begin{aligned}
&\mathbf{E}^{Q_i}(\Theta_T \cdot S_T) = \mathbf{E}^{Q_i}(\Theta_1 \cdot S_0) = \Theta_1 \cdot S_0,\ \forall \Theta \in \mathcal{T}(\mathcal{A}),\ i = 1, 2\\
\Rightarrow &\mathbf{E}^{Q_1}(\Theta_T \cdot S_T) = \mathbf{E}^{Q_2}(\Theta_T \cdot S_T),\ \forall \Theta \in \mathcal{T}(\mathcal{A})\\
\Rightarrow &Q_1 = Q_2 \text{ (since } R = RV(\Omega))
\end{aligned}$$

Only if part: Suppose, $Q \in \mathcal{M}(\mathcal{A})$ and $R \neq RV(\Omega)$. Then, $\exists\, \xi \in R^{\perp}$ such that, $\xi \neq 0$. Observe that, $\vec{1} \in R$, since we can buy and hold the risk-free asset a_0. This implies, $\langle \xi, \vec{1} \rangle = 0$.

Now, $\exists\, c > 0$ such that, $Q' := Q + c\xi P > 0$. Since, $\langle \xi, \vec{1} \rangle = 0$, we have,

$$\int_{\Omega} dQ' = \int_{\Omega} dQ + c \int_{\Omega} \xi dP = 1 + c\langle \xi, \vec{1} \rangle = 1.$$

Hence, Q' is a probability measure on Ω.

Since, $Q \in \mathcal{M}(\mathcal{A})$, we have, $\mathbf{E}^Q(\Theta_T \cdot S_T) = \Theta_1 \cdot S_0$. Observe that,

$$\begin{aligned}\mathbf{E}^{Q'}(\Theta_T \cdot S_T) = & \int_\Omega \Theta_T \cdot S_T \, dQ + c \int_\Omega \Theta_T \cdot S_T \, \xi \, dP \\ =& \mathbf{E}^Q(\Theta_T \cdot S_T) + c \, \langle \Theta_T \cdot S_T, \xi \rangle \\ =& \Theta_1 \cdot S_0 \text{ since } \xi \perp \Theta_T \cdot S_T \in R\end{aligned}$$

This proves that Q' is a martingale measure. But, clearly $Q \neq Q'$. This is a contradiction to the fact that $\mathcal{M}(\mathcal{A})$ is singleton. Q.E.D.

Pricing with martingale or risk neutral measure is rather prevailing in the financial derivative market. While this method provides us a theoretical price of the financial derivative it does not tell us what to do when the market price deviates from the theoretical price. Theoretically one should solving the original utility optimization problem to derive the optimal trading strategy which in this case will be an arbitrage. This means in practice we should (a) solving (5.5) with the theoretical price to get a replicating portfolio; (b) with the difference of the market price and the theoretical price an arbitrage trading strategy exists; and (c) using all the available leverage. Item (c) clearly reminds us the recent financial crisis and the over leverage associated with it. What has been over looked? The sensitivity!

5.4 Sensitivity

Once the Black and Scholes replicating portfolio method and the Cox-Ross risk neutral pricing method are put in the perspective of a utility optimization problem and its dual, sensitivity to model perturbation becomes a natural issue. Experience tells us that optimal solutions in many cases are sensitive to model perturbations. It turns out that the replicating portfolio is exactly one of those that are quite sensitive. To illustrate let us look at the a slight generalization of the one period model in Section 5.1. Again consider a stock index whose spot price is standardized to $S_0 = 1$ and an at money call option with a strike price 1. We assume a one period model in which the stock index and its options can only be traded now when $t = 0$ and at the maturity (M) of the option represented by $t = 1$. Assumes that a market agent perceives that S_1 can take only two values $1 + a$ or $1 - b$, with $a, b > 0$. Then the perceived value of the stock index and its option at maturity is represented in the following table:

$$
\begin{array}{cccc}
\text{Stock} & \text{Maturity} & \text{Option} & \text{Maturity} \\
\begin{tikzcd}[row sep=small] & 1+a \\ 1 \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 1-b \end{tikzcd} & & \begin{tikzcd}[row sep=small] & a \\ c \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 0 \end{tikzcd} &
\end{array}
\tag{5.10}
$$

To this agent the option price should be $c = ab/(a + b)$. When the market option price is deviated from c to, say $c + dc$ with $dc > 0$, the arbitrage position should be sell one call option at $c + dc$ and buy $\frac{a}{a+b}$ shares of stock at 1.

Now assume, as often occurs in the real world, that the actual price of the stock index at $t = 1$ is also somewhat deviated from the theoretical model as represented in (5.11).

$$
\begin{array}{cccc}
\text{Stock} & \text{Maturity} & \text{Option} & \text{Maturity} \\
\begin{tikzcd}[row sep=small] & 1+a+da \\ 1 \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 1-b-db \end{tikzcd} & & \begin{tikzcd}[row sep=small] & a+da \\ c+dc \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & 0 \end{tikzcd} &
\end{array}
\tag{5.11}
$$

Then the actual payoff of the above 'arbitrage' trading strategy would be as described in (5.12)

$$
\begin{array}{ccc}
\text{Cost} & \text{Payoff} & \text{Percentage gain} \\
\begin{tikzcd}[row sep=small] & \frac{a(1-b)+ada}{a+b} \\ \frac{a(1-b)}{a+b} - dc \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & \frac{a(1-b)-adb}{a+b} \end{tikzcd} & & \begin{matrix} \frac{ada+(a+b)dc}{a(1-b)-(a+b)dc} \\ \\ \frac{-adb+(a+b)dc}{a(1-b)-(a+b)dc} \end{matrix}
\end{array}
\tag{5.12}
$$

Since typically $|da|, |db| > dc$ the perceived safe 'arbitrage' may turn out to be a pure losing position in both scenarios. If one leverage heavily on such an 'arbitrage' the result will be disastrous. Unfortunately, huge amount of highly leveraged 'safe' financial derivatives turn into losing positions in many financial institutes was exactly what happened during the 2008 financial crisis. It is time to explore alternative pricing and trading methods for financial derivatives.

5.5 A robust option pricing and trading strategy

Here we discuss a robust option pricing and trading method based on private information studied in [10, 11, 49, 51] with test on historical market data.

Real financial markets often have trends. This phenomenon is well known to practitioners (see [1, 12, 34]) and taking advantage of using various method such as trend following and contra-trend trading. Theoretically justification of the trends in markets using information asymmetry cab found in [18, 19]. Assume that in our financial market model one stock index is the only risky asset and one can either buy or write a at the money call option on the stock index. We use a random variable x to represent the discounted percentage gain of using the private information, say on the trend of the market to trade and assume that $\mathbf{E}(x) > 0$. Let p be the price of the option. Then the percentage gains of using the same method to buy or write call option are $C(p) = \frac{x^+ - p}{p}$ and $W(p) = \frac{p - x^-}{1-p}$, respectively. A heuristic argument in [51] show that taking only position of the stock index, buy a call or write a call using the trading strategy related to our private information is a robust strategy (may not be optimal). The idea is to establish a criterion so that we can switch among the three. We use the indicator in [50, 51] based on utility maximization:

$$r(x) = \sup\{\mathbf{E}[u(1+sx)] \mid s \in [0, \infty)\},$$

where u is a utility function that satisfies conditions (u1)-(u4), s represents the share invested in the index and the range $(0, \infty)$ implies trade on margin is allowed. The decision on what position to take among the index itself, buying the call or writing the call is based comparing $r(x), r(C(p))$ and $r(W(p))$. Figure 5 is a typical configuration of these three quantities as functions of p. It turns out that picture don't lie. What we observe in the picture can indeed be rigorously justified. In general, there exist two critical call option premiums $p^C \le p^W$ determined by the intersections of $r(x)$ with $r(C(p))$ and $r(W(p))$, respectively. When the option premium $p < p^C$, buying call options will improve the investment system and when $p > p^W$, writing covered call options will improve the investment system. For any option premium p that belongs to the option pricing interval $[p^C, p^W]$ the original investment system gives the best result. The fact that, in general, $p^C < p^W$ indicates that buying calls and writing calls should be treated as different alternatives and should be priced differently. Detailed discussions including estimates for those thresholds can be found in [51].

To implement this strategy in real financial market, we choose to use the log utility due to many of the advantages alluded to in Section 4.4. However, as pointed out in Section 4.4. the log utility also has a serious

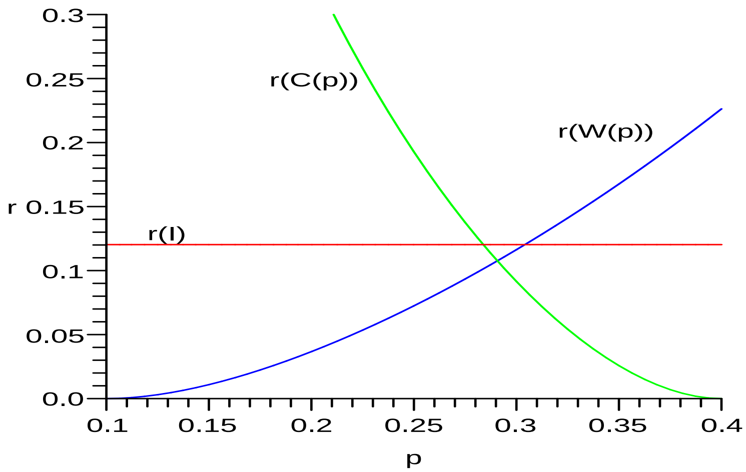

Figure 5: Critical option premiums

shortcoming – too sensitive to model perturbation. To help mitigate the risk we add additional risk control based on a restriction on the conditional value at risk (CVaR), a convex risk measure introduced in [36]. The 'private' information is a classical trend following system signaled by a short 40 day price moving average cross a long 200 day price moving average. We refer to [10, 11] for details of the implementation. We tested using the SP500 index and the comparison with the buy and hold the index is shown in Figures 6 and 7 using risk control of CVaR less than 0.1 and 0.05, respectively.
Trends in the markets are well known to practitioners [1, 34, 39] and are justified theoretically using information asymmetry in [18, 19]. Theoretical justifications of using trend following methods in the equity markets can be found in [8] where optimal strategy for a regime switching model is also discussed. The market tests in [10, 11] deliberately used a well known trend following method. This way the private information is not truly private and, therefore, direct gain from this trend following method is limited. Despite that our method still significantly outperform the index. This highlights the market price for option (which largely follow the paradigm of Black-Scholes) is not efficient. Design trading strategy by maximizing utility function has a long history going back at least to [45]. Kramkov and Schachermayer [26]

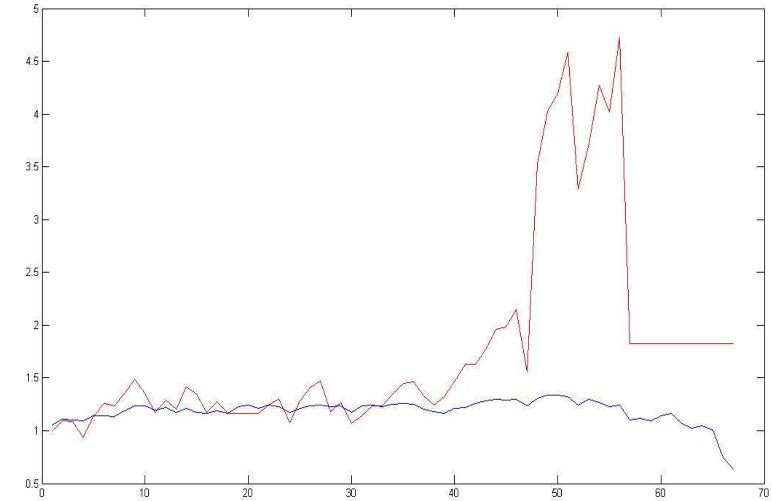

Figure 6: Robust trading method SP500 market data test

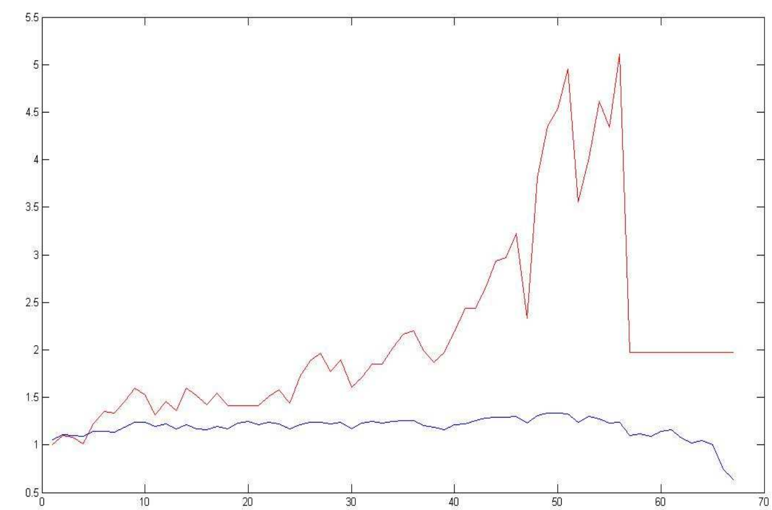

Figure 7: Robust trading method SP500 market data test

is a good source for the history of this method and recent development. On the other hand, Föllmer and Leukert suggested quantile and efficient hedging in [13, 14]. The idea is to trade the financial derivatives and their underlying assets to minimize a risk measure. This line of research was generalized by Rudloff in [38]. Vince [47, 48] emphasized the need to trade-off between utility and risk and his method of trade-off is to use a fraction of the optimal leverage of a growth portfolio. The tests alluded to above shows the advantage of combining a utility function and a risk measure with different features.

Clearly, using different private information, utility functions and risk measures will lead to different pricing and trading methods for options and more general financial derivatives. Such idea of pricing financial derivatives using private information and criteria has a long history and actually predates the work of Black and Scholes. One influential account of such a pricing method for warrant is Thorp and Kassouf's book [45]. However, during the past several decades, the replicating portfolio and risk-neutral measure pricing became dominant. The wide acceptance of such an uniform pricing greatly facilitated the trading of the financial derivatives and increased the liquidity of financial derivative products. This is one of the most important reasons that leads to the unprecedented prosperity of the financial derivative products prior to the 2008 financial derivative market crash. Now on hinder sight the uniform pricing of the derivatives and the consequent boom of the derivative markets seem to be important factors that lead to the financial crisis. With the memory of the 2008 financial crisis still fresh, it is useful to rethink about the merits of alternative mechanisms for pricing and trading the financial derivatives. The example provided in this section and other related methods commented above indicate that convex analysis is likely to play a central role in such an endeavor.

6 Consumption based asset pricing model

The pricing mechanism we discussed so far are all relative in the sense that the price of a particular financial asset is determined by comparing them with other assets whose price is assumed to be observable from the market or follows a certain model. What ultimately determines the price? Economists will tell us that it is the interaction of supply and demand. This type of models belong to the realm of general equilibrium theorem. While the idea can be traced back Adam Smith, a mathematical model for a general equilibrium was first discussed by L. Walras and shaped and proved rigorous by Kenneth Arrow and Gerald Debrue [21].

While theoretically the Arrow-Debrue model can be used to determine all the prices in a competitive economy in reality the model is too general to calibrate making it mainly a theoretical framework. However, the idea of price discovery using general economic equilibrium has been profoundly influential and has been adapted to numerous special cases where additional structures are available. Lucas' model is one of the very influential in which convex analysis plays a crucial role [29].

In the Lucas model the price process S_t of various assets is not given. Rather it is derived from focusing on the equilibrium of consumption and production. Lucas adopt a simplify the economic model that has only a single consumption goods and one consumer who represents the behavior of the aggregated demand inside the economy. The consumer tries to maximizing the utility of the total consumption

$$\mathbf{E}\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\} \tag{6.1}$$

where c_t is a stochastic process of consumption, u is a utility function and β is a factor that discounts the consumption in the future. The consumption good is produced by M different producers $\mathcal{A} = \{a^1, a^2, \ldots, a^M\}$ whose output on period t is represented by a vector $y_t = (y_{1t}, \ldots, y_{nt})$ and assumed to follow a Markov process defined by its transition function

$$F(y', y) = Prob(y_{t+1} \leq y' \mid y_t = y).$$

Output is perishable so that the consumption must satisfy the constraint

$$0 \leq c_t \leq \sum_{n=1}^{M} y_{nt}. \tag{6.2}$$

The ownership of each producer is represented by one share of equity which entitles the share holder the output of the producer in proportion to the share. As before the price processes of the (risky) assets and the trading strategy of the consumer are denoted by $\hat{S} = (S^1, \ldots, S^M)$ and $\hat{\Theta}$, respectively.

At any decision time $t = \bar{t}$ the consumer's problem is

$$\begin{aligned} v(\hat{\Theta}_{\bar{t}}, y_{\bar{t}}) &= \max \mathbf{E} \left\{ \sum_{t=\bar{t}}^{\infty} \beta^t u(c_t) \right\} && (6.3) \\ \text{subject to} \quad & c_{t+1} + \hat{\Theta}_{t+1} \cdot \hat{S}_{t+1} \leq (\hat{S}_{t+1} + y_{t+1}) \cdot \hat{\Theta}_t, t = \bar{t}, \bar{t}+1, \ldots \\ & c_t \geq 0. \end{aligned}$$

Since the output process is Markov and stationary, the optimal value function v is independent of the decision time $\bar{t}$. Using θ and y to denote the generic portfolio and output then the value function $v(\theta, y)$ satisfies the dynamic programming equation

$$\begin{aligned} v(\theta, y) &= \max_{c,\theta'} \mathbf{E} \left\{ u(c) + \beta \int v(\theta', y') dF(y', y) \right\} && (6.4) \\ \text{subject to} \quad & c + \theta' \cdot \hat{S} \leq (\hat{S} + y) \cdot \theta, c \geq 0. \end{aligned}$$

In equilibrium the consumer should consume all the goods and own all the shares of the producers. This leads to the following definition:

Definition 6.1. *An equilibrium is a pair of continuous functions $S(y)$ and $v(\theta, y)$ satisfying the dynamic programming equation (6.4), $c = \sum_{n=1}^{M} y_n$ and $\theta = \vec{1} = (1, 1, \ldots, 1)$.*

At equilibrium the optimal value function $v(\vec{1}, y)$ satisfies the equation

$$v(\vec{1}, y) = u(y \cdot \vec{1}) + \beta \int v(\vec{1}, y') dF(y', y). \tag{6.5}$$

It is easy to check that

$$v \to u(y \cdot \vec{1}) + \beta \int v(\vec{1}, y') dF(y', y)$$

is a contraction. Thus (6.5) uniquely defines v. We now turn to find the equilibrium price $\hat{S}$. First let $T : C(R^M \times R^M)^+ \to C(R^M \times R^M)^+$ be an operator such that equation (6.4) is equivalent to $v = Tv$.

Lemma 6.2. *The dynamic programming equation (6.4) has a unique solution which is nondecreasing concave function of θ for every y.*

Proof. It is easy to verify that, for any $v \geq w$, $Tv \geq Tw$ and, for any, constant α, $T(v+\alpha) = Tv + \beta\alpha$. Observing that $w \leq v + \|v - w\|$ for the sup norm $\|\cdot\|$ we have

$$Tw \leq Tv + \beta\|v - w\|.$$

Switch the position of v and w we have

$$Tv \leq Tw + \beta\|v - w\|.$$

Thus, T is a contraction and, therefore, equation (6.4) has a unique solution.

Observing that Tu is an increasing function of θ for any u so is $v = Tv$. Finally, if u is concave in θ then so is Tu. It follows that $v = \lim_{n\to\infty} T^n 1$ is also concave in θ. Q.E.D.

When u is smooth it was shown in [29], using elementary calculus, that $v(\cdot, y)$ is smooth and

$$v_\theta(\theta, y) = u'(c)[y + S(y)]. \tag{6.6}$$

Since the right hand side of the dynamic programming equation (6.4) is a concave optimization problem, the first order optimality condition is both necessary and sufficient for the optimal solution. This leads to

Theorem 6.3. *The equilibrium price of the equity $S(y)$ is the unique solution of*

$$u'(\sum_{n=1}^{M} y_n)S(y) = \beta \int u'(\sum_{n=1}^{M} y_n')(y' + S(y'))dF(y', y). \tag{6.7}$$

Proof. It is easy to see that the right hand side of (6.3) is a contraction for $u'(\sum_{n=1}^M y_n)S(y)$ and, therefore, equation (6.3) uniquely determines $u'(\sum_{n=1}^M y_n)S(y)$. Since u is strictly increasing, $u' > 0$ and $S(y)$ is uniquely determined. Q.E.D.

The methods used by Lucas can be extended to handle situations involving bonds and to deal with the term structure of interest rates. For example, Duan and Zhu [9] used the Lucas model to explore the term structure of interest rates when the consumer are subject to structure consumption commitments and derived equilibrium solutions under verifiable conditions. The model in [9] preserves the feature of the Lucas' original model in using one representative consumer but involved bonds of different maturities to explore the impact of consumption commitments to the term structure of interest rates. The bonds with different maturities complicate the structure of the problem and the equilibrium in this model has to be derived by solving several iterative equations. Nevertheless, the equilibrium price can be derived through a numerical procedure and the simulation results leads to qualitative properties that are consistent with intuitions. Judd, Kubler and Schmedders in [23, 24, 27] extended the Lucas' model in a different direction. The emphasis in those studies are issues related to Perato inefficiency of the competitive equilibrium, investors portfolio separation and trading volume of the bonds. The model in [23, 24, 27] assumes the agents can both purchase and issue bonds and the quantity of bonds net to 0. This necessitates to consider multiple agents.

7 Conclusion

Using a uniform framework and convex analysis we provided a concise tour of some of the important results in both theory and practice of financial mathematics. This is not an attempt for a comprehensive review of the applications of convex analysis in the area of finance. Nevertheless, the examples presented here, I hope, will provide enough evidence to suggest that convex analysis is fundamental in dealing with financial problems and, reciprocally, applications in the financial area raise interesting challenges to researchers in convex and variational analysis.

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