Implicit Multifunction Theorems

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Abstract. We prove a general implicit function theorem for multifunctions with a metric estimate on the implicit multifunction and a characterization of its coderivative. Traditional open covering theorems, stability results, and sufficient conditions for a multifunction to be metrically regular or pseudo-Lipschitzian can be deduced from this implicit function theorem. We prove this implicit multifunction theorem by reducing it to an implicit function/solvability theorem for functions. This approach can also be used to prove the Robinson-Ursescu open mapping theorem. As a tool for this alternative proof of the Robinson-Ursescu theorem we also establish a refined version of the multidirectional mean value inequality which is of independent interest.

Key Words. Nonsmooth analysis, subdifferentials, coderivatives, implicit function theorem, solvability, stability, open mapping theorem, metric regularity, multidirectional mean value inequality.

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1 Introduction

In this paper we prove a general implicit multifunction theorem for the inclusion

$$0 \in F(x,p) \tag{1}$$

where $F : X \times P \to 2^Y$ is a multifunction, X and Y are Banach spaces and P is a metric space. Variants of this general theorem have been studied under different names and in different contexts (see for example [1, 2, 5, 8, 9, 11, 18, 19, 22, 26, 27, 28, 37, 38, 43, 48, 51, 55, 56, 57, 59]). The aim of this paper is to present an unified and streamlined approach to the derivation of a general implicit multifunction theorem based on a recent multidirectional mean value inequality [16]. This means to make the vast body of related results accessible to non-experts in nonsmooth analysis who are ready to accept a few basic results. A novel feature of our result is that we characterize the coderivative of the implicit multifunction in terms of F. In this sense our implicit multifunction theorem corresponds to the classical implicit function theorem which gives not only the existence of the implicit function but also a formula for its derivative. Related formulae for coderivative of implicit multifunctions were computed for important case of the so-called constaint systems and generalized equation/variational inequalities by Mordukhovich [44, 45, 46] (see also [47] for infinite dimensional extensions). One-sided estimate for an implicit multifunction in terms of outer graphical derivatives was discussed in Levy [39] and Levy and Rockafellar [40] (see also [3]). Moreover, our infinitesimal conditions for the existence of the implicit multifunction require one to check only coderivatives of projections or 'approximate projections'. This condition is weaker than prevailing coderivative conditions, which typically require checking the coderivative at all points. As a useful tool we established a representation formula for the Fréchet subgradient of the lower semicontinuous envelope of a marginal function under very general assumptions. This result is motivated by [6, 35, 61] and is interesting by itself. As an application we deduce sufficient infinitesimal conditions for metric regularity, open covering, Aubin continuity for multifunctions and Ioffe type global covering result. We also discuss relationship between stability of the multifunction and the infinitesimal regularity conditions in the implicit multifunction theorem. Finally we show our approach can also be used to give an alternative proof of the Robinson-Ursescu open mapping theorem. In this argument, a refined multidirectional mean value inequality for quasidifferentiabl functions (see [54]) is needed. This is another result of independent interest.

The study of implicit function theorems has a long history. When F is single-valued, (1) becomes

$$F(x, p) = 0. \tag{2}$$

This is the setting of the classical implicit function theorem. In particular, when the image of F is in a finite dimensional space, (2) becomes a system of equations with a parameter p,

$$f_i(x, p) = 0,\ i = 1, ..., m. \tag{3}$$

We can view the problem of finding the implicit function defined by (2) as the solvability problem of (3), which is important in constrained optimization. When $x \in R^n$ and $m = n$, the classical implicit function theorem tells us that if $(\bar{x}, \bar{p})$ is a solution to (3) and the partial gradients of the f_i, $i = 1, ..., m$ with respect to x are continuous and linearly independent at $(\bar{x}, \bar{p})$, then for any p near $\bar{p}$, there exists a unique solution $x = g(p)$ of (3); further, the function g is continuous at p. This theorem applies when the number of equations m is strictly less than the number of unknown variables n. In such a case the solutions corresponding to p form, in general, a nonempty set $G(p)$, rather than a single point, which is also continuous at $\bar{p}$. However, the classical implicit function theorem cannot handle the solvability of a system of inequalities, such as

$$f_i(x, p) \leq 0, i = 1, .., m. \tag{4}$$

Such systems are equally important in dealing with constrained minimization problems. Here we already perceive the need for dealing with multifunctions. Actually if $F(x, p) = \{(y_1, ..., y_I) : y_i \geq f_i(x, p), i = 1, ..., I\}$, then (4) becomes (1) as pointed out by Robinson in [56]. The problem of solving (1) also arises in problems related to variational inequalities, open covering properties, metric regularity and many other areas. In fact, implicit multifunctions have been studied under different names for the aforementioned problems. Methods and concepts for dealing with these types of problems, in particular variational arguments and the concept of coderivatives, have been developed systematically.

The general approach adopted in this paper is based on sufficient infinitesimal conditions for the solvability of the scalar inequality

$$f(x, p) \leq 0, \tag{5}$$

where f is a nonsmooth function. This approach and the related variational argument used for deriving infinitesimal conditions for solvability and for proving a nonsmooth implicit function theorem was inaugurated by Clarke [11, 12]. We mentioned before that there are many other related results. One way to put these related work in perspective is to observe that a multidirectional mean value inequality is one of the several equivalent tools in the calculus of subdifferentials. The other major equivalent tools are a sum rule, an extremal principle and a nonlocal sum rule (see [34, 36, 66]). All of them are based on the variational principles [7, 25]. Thus, a proof of an implicit multifunction theorem can also be derived by starting from any of those basic calculus tools or by using the variational principle directely. Various similar variational approaches were used by Ioffe [28] to established metric regularity properties for systems of equations with nonsmooth data, by Aubin [1] and Aubin and Frankowska [2, 3] to discuss open covering theorems and solvability, by Rockafellar [59] to discuss Lipschitz properties of multifunctions, by Borwein [5] to obtain conditions for metric regularity for a general system of inequalities, and by Ledyaev to obtain certain variants of the nonsmooth implicit multifunction theorem for Banach spaces in [37] and for metric spaces in [38]. In the latter, infinitesimal conditions were formulated in terms of derivate constructions analogous to directional derivatives. Related results were also developed in Azé et al [4] and Penot [50, 52] emphasizing the role of tangent cones. Mordukhovich [43] and Mordukhovich and Shao [48] give necessary and sufficient coderivative conditions for open coverings, metric regularity and Aubin continuity for multifunctions and show that these properties are equivalent. Sufficient conditions for those properties can easily be deduced from our implicit multifunction theorem. The converse is also possible but that need additional assumptions, e.g., both X and P are smooth Banach spaces. Therefore, in our opinion, an implicit multifunction theorem is a more general result.

In this paper, infinitesimal conditions for the existence of an implicit multifunction are expressed in terms of coderivatives introduced by Mordukhovich [41, 42]. The key in our approach is to establish the relation between the subdifferential infinitesimal condition for $f(x,p) = d(0, F(x,p))$ and the coderivative infinitesimal condition for F. Here, Ioffe and Penot's result on the subdifferential of marginal functions is relevant [35]. This method has an additional benefit. It shows clearly that the infinitesimal condition is only relevant for those points in $F(x,p)$ with a norm close to $d(0, F(x,p))$. This refinement of the infinitesimal condition is significant as shown by examples in Section 3. We state all the results in terms of the Fréchet subdifferential. Recent results of Ioffe [34] and Lassonde [36] indicate that generalizations to abstract subdifferentials are possible.

Implicit multifunction theorems have also been discussed by using approximations. For research in this direction we refer the readers to Dontchev and Hager [23, 24], Ioffe [29], Robinson [58], Sussmann [60], Warga [63, 64] and the references therein.

We arrange the paper as follows: In section 2 we set the notation and recall and prove some preliminary results that are needed in what follows. We discuss implicit multifunction theorems in Section 3. Sections 4 and 5 are devoted to the relationships with open covering theorems, conditions for metric regularity, pseudo-Lipschitz properties of multifunctions, and stability. In section 6 we apply our method to convex multifunctions for proving the Robinson-Ursescu open mapping theorem [56, 62]. We also prove a refined version of the multidirectional mean value inequality.

2 Notation and Preliminaries

Let X be a Banach space with continuous real dual X^*. We denote by 2^X the collection of all subsets of X and use $\bar{R}$ to denote the extended real line $R \cup \{\infty\}$. For any $x \in X$ and $r > 0$, we denote the closed ball centered at x with radius r by $B_r(x)$ or $B(x,r)$ and we denote the closed unit ball in X by B_X. Let x be an element of X and S a subset of X; the distance between x and S is $d(x,S) := \{\|x - s\| : s \in S\}$. We denote by *cone* $S := \cup_{k>0} kS$ the cone generated by S. Let us recall the definitions of the Fréchet subdifferential and normal cone. Recall that a *bump* function is a bounded real-valued function which has a bounded nonempty support.

Definition 2.1 *Let X be a Banach space, let $f : X \to \bar{R}$ be a lower semicontinuous function and S a closed subset of X. We say f is Fréchet-subdifferentiable and x^* is a Fréchet-subderivative of f at x if there exists a C^1 function g such that $g'(x) = x^*$ and $f - g$ attains a local minimum at x. We denote the set of all Fréchet-subderivatives of f at x by $\partial_F f(x)$. We define the Fréchet-normal cone of S at x to be $N(x,S) := \partial_F \delta_S(x)$ where δ_S is the indicator function of S (i.e., $\delta_S(x) = 0$ for $x \in S$ and ∞ otherwise).*

It is easy to see that $N(x,S) = \text{cone}\ \partial_F d(x,S)$. When a function f has more than one 'variable', say $f = f(x,y)$, we use $\partial_{F,x}$ to signify the 'partial subdifferential' of f with respect to the variable x. If function f is convex then it is easy to see that $\partial_F f(x)$ coinsides with a classical subdifferential

$\partial f(x)$ from convex analysis. In the case $f(x) = \|x\|$ and $\|x\| > 0$ the inclusion $x^* \in \partial f(x)$ implies $\|x^*\| = 1$.

Let X and Y be metric spaces and let $F : X \to 2^Y$ be a multifunction. We denote the graph of F by Graph $F := \{(x, y) : y \in F(x)\}$. We say F has a closed graph when Graph F is closed. Recall that F is upper semicontinuous at x if for any open set O contains $F(x)$ there exists an open neighborhood V of x such that $x' \in V$ implies that $F(x') \subset O$. A multifunction F is lower semicontinuous at x if for any $y \in F(x)$ and any open neighborhood W of y there exists an open neighborhood V of x such that $x' \in V$ implies that $F(x') \cap W \neq \emptyset$. Next we recall the concept of the Fréchet coderivative.

Definition 2.2 *Let X and Y be Banach spaces and let $F : X \to 2^Y$ be a multifunction with closed graph and $y \in F(x)$. If*

$$(x^*, -y^*) \in N((x, y), \mathrm{Graph}(F))$$

then we say x^* is a Fréchet coderivative of F at (x, y) corresponding to y^*. We denote the set of coderivatives of F at (x, y) corresponding to y^* by $D^*F(x; y)(y^*)$.

It is easy to see that if F is a Fréchet smooth single valued mapping then $D^*F(x, F(x))(y^*) = (F'(x))^*y^*$. Thus, the Fréchet coderivative is an extension of the adjoint of the Fréchet derivative.

The main tools that we use in this paper are a smooth variational principle (see [7, 21]), a fuzzy sum rule (see [30, 33, 9]) a multidirectional mean value theorem (see [16, 65]) and a decrease principle (see [17]). For convenience, we recall these results in the forms that will be used in this paper below. We start with the smooth variational principle.

Theorem 2.3 (Smooth Variational Principle) *Let X be a Banach space with a Fréchet-smooth Lipschitz bump function and let $f : X \to \bar{R}$ be a lower semicontinuous function bounded from below. Then there exists a constant $\alpha > 0$ (depend only on X) such that for all $\varepsilon \in (0, 1)$ and for any u satisfying*

$$f(u) < \inf_X f + \alpha\varepsilon^2,$$

there exists a Fréchet-differentiable function g on X and v in X such that

(i) *The function*

$$x \to f(x) + g(x)$$

attains a global minimum at $x = v$,

(ii)

$$\|u - v\| < \varepsilon,$$

(iii)

$$\max(\|g\|_\infty, \|g'\|_\infty) < \varepsilon.$$

Next we state the fuzzy sum rule for Fréchet subdifferentials.

Theorem 2.4 *Let X be a Banach space with a Fréchet-smooth bump function and let $f_1, ..., f_N : X \to \bar{R}$ be lower semicontinuous functions. Suppose that all but one of $f_1, ..., f_N$ are locally Lipshcitz around $\bar{x}$ and $\sum_{n=1}^N f_n$ attains a local minimum at $\bar{x}$. Then, for any $\varepsilon > 0$, there exist $x_n \in \bar{x}+\varepsilon B$ and $x_n^* \in \partial_F f_n(x_n), n = 1, ..., N$ such that $|f_n(x_n) - f_n(x)| < \varepsilon, n = 1, 2, ..., N$ and*

$$\| \sum_{n=1}^N x_n^* \| < \varepsilon.$$

To state the multidirectional mean value inequality we denote the convex hull of a point x and a set Y by $[x, Y] := \{x + t(y - x) : y \in Y, \, t \in [0, 1]\}$.

Theorem 2.5 (Multidirectional Mean Value Inequality) *Let X be a Banach space with a Fréchet-smooth Lipschitz bump function, let Y be a nonempty, bounded, closed and convex subset of X, let $x \in X$ and let $f : X \to \bar{R}$ be a lower semicontinuous function. Suppose that, for some $h > 0$, f is bounded below on $[x, Y] + hB_X$ and*

$$\lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f(y) - f(x) > r.$$

Then, for any $\varepsilon > 0$, there exist $z \in [x, Y] + \varepsilon B$ and $z^* \in \partial_F f(z)$ such that

$$r < \langle z^*, y - x \rangle \text{ for all } y \in Y.$$

Further, we can choose z so that

$$f(z) < \lim_{\eta \to 0} \inf_{[x,Y]+\eta B_X} f + |r| + \varepsilon.$$

The decrease principle follows easily from the multidirectional mean value theorem [17].

Theorem 2.6 (Decrease Principle) *Let X be a Banach space with a Fréchet-smooth Lipschitz bump function, let $f : X \to \bar{R}$ be a lower semicontinuous function bounded from below and let $r > 0$. Suppose that, for any $x \in B_r(\bar{x})$, $\xi \in \partial_F f(x)$ implies that $\|\xi\| > \sigma > 0$. Then*

$$\inf_{x \in B_r(\bar{x})} f(x) \leq f(\bar{x}) - \sigma r.$$

We will also need the following basic result on the subdifferential of the marginal (optimal value) function

$$f(x) := \inf_{y \in Y} \varphi(x, y)$$

where φ is a lower semicontinuous function. For related results see Borwein and Ioffe [6], Ioffe and Penot [35] and Thibault [61]. Actually, we will discuss, in a more general form, the subdifferential of the lower semicontinuous envelope of the marginal function. Recall that for a function $f : X \to \bar{R}$ the lower semicontinuous envelope of f is defined by

$$\underline{f}(x) := \lim_{h \to 0+} \inf_{\|y-x\|<h} f(y).$$

Theorem 2.7 (Subdifferential of Marginal Functions) *Let X be a Banach space with Fréchet-smooth Lipschitz bump functions, let Y be an arbitrary set, let function $\varphi(\cdot, y)$ be lower semicontinuous for any $y \in Y$ and let $\xi \in \partial_F \underline{f}(x)$ ($\xi \in \partial_F f(x)$). Then, for any $\varepsilon > 0$, there exist $(x_\varepsilon, y_\varepsilon)$ and $\xi_\varepsilon \in \partial_{F,x} \varphi(x_\varepsilon, y_\varepsilon)$ such that $\|x - x_\varepsilon\| < \varepsilon$, $|\underline{f}(x) - \underline{f}(x_\varepsilon)| < \varepsilon$, ($|f(x) - f(x_\varepsilon)| < \varepsilon$,)*

$$\varphi(x_\varepsilon, y_\varepsilon) < \underline{f}(x_\varepsilon) + \varepsilon < f(x_\varepsilon) + \varepsilon, \tag{6}$$

and

$$\|\xi_\varepsilon - \xi\| < \varepsilon. \tag{7}$$

If, in addition, Y is a Banach space with Fréchet-smooth Lipschitz bump functions and φ is lower semicontinuous on $X \times Y$ then there exists an element of the joint Fréchet subdifferential of φ, $(\xi_\varepsilon, \zeta_\varepsilon) \in \partial_F \varphi(x_\varepsilon, y_\varepsilon)$ such that also

$$\|\xi_\varepsilon - \xi\| < \varepsilon, \quad \|\zeta_\varepsilon\| < \varepsilon. \tag{8}$$

Proof. We will give the full proof for the case when Y is a Banach space with Fréchet-smooth Lipschitz bump functions and indicate the modification needed for the case when Y is an arbitrary set in the end.

Note that f is lower semicontinuous at x when $\partial_F f(x) \neq \emptyset$. Moreover, it is easy to check that $\partial_F f(x) \subset \partial_F \underline{f}(x)$. Thus we need only prove the theorem when $\xi \in \partial_F \underline{f}(x)$. Let g be a C^1 function such that $g'(x) = \xi$ and $\underline{f} - g$ attains a minimum 0 at x over $B_r(x)$ for some $r \in (0,1)$. Let $\alpha > 0$ be the constant associated with X in the Smooth Variational Principle of Theorem 2.3. Let $\varepsilon \in (0, r)$ be an arbitrary positive number. Choose $\eta < \min(\varepsilon/5, \sqrt{\varepsilon/5\alpha})$ such that $x' \in B_\eta(x)$ implies that $\underline{f}(x') \geq \underline{f}(x) - \varepsilon/5$, $\|g'(x') - g'(x)\| < \varepsilon/2$ and $\|x' - x''\| < \eta$ implies that $|g(x') - g(x'')| < \varepsilon/5$. Taking $z \in B_{\eta/2}(x)$ close enough to x so that $f(z) - g(z) < \underline{f}(x) - g(x) + \alpha\eta^2/8$, one can choose $y \in Y$ satisfying

$$\begin{aligned} \varphi(z,y) - g(z) &< f(z) - g(z) + \alpha\eta^2/8 \\ &< \underline{f}(x) - g(x) + \alpha\eta^2/4 \\ &\leq \inf_{(u,v)\in B_r(x)\times Y} (\varphi(u,v) - g(u)) + \alpha\eta^2/4. \end{aligned} \tag{9}$$

Applying the Smooth Variational Principle of Theorem 2.3 to the function $(u,v) \to \varphi(u,v) - g(u)$ we obtain $(x_\varepsilon, y_\varepsilon) \in B_{\eta/2}(z,y) \subset B_\eta(x,y)$ and a C^1 function h such that $\max(\|h\|_\infty, \|h'\|_\infty) < \eta$, and the function

$$(u,v) \to \varphi(u,v) - g(u) + h(u,v)$$

attains its minimum at $(u,v) = (x_\varepsilon, y_\varepsilon)$. Then

$$(g'(x_\varepsilon) - h'_x(x_\varepsilon, y_\varepsilon), -h'_y(x_\varepsilon, y_\varepsilon)) \in \partial_F \varphi(x_\varepsilon, y_\varepsilon)$$

and (8) follows with $\xi_\varepsilon = g'(x_\varepsilon) - h'_x(x_\varepsilon, y_\varepsilon)$ and $\zeta_\varepsilon = -h'_y(x_\varepsilon, y_\varepsilon)$. It remains to verify inequality (6) and $|\underline{f}(x) - \underline{f}(x_\varepsilon)| < \varepsilon$, $(|f(x) - f(x_\varepsilon)| < \varepsilon)$:

$$\begin{aligned} \varphi(x_\varepsilon, y_\varepsilon) &\leq \varphi(z,y) + [g(x_\varepsilon) - g(z)] + h(z,y) - h(x_\varepsilon, y_\varepsilon) \\ &\leq \underline{f}(x) + \alpha\eta^2 + |g(x_\varepsilon) - g(z)| + 2\|h\|_\infty \\ &< \underline{f}(x_\varepsilon) + \varepsilon/5 + \alpha\eta^2 + |g(x_\varepsilon) - g(z)| + 2\|h\|_\infty < f(x_\varepsilon) + \varepsilon. \end{aligned}$$

Since $\underline{f}(x_\varepsilon) \leq f(x_\varepsilon) \leq \varphi(x_\varepsilon, y_\varepsilon)$ we have $|\underline{f}(x) - \underline{f}(x_\varepsilon)| < \varepsilon$, $(|f(x) - f(x_\varepsilon)| < \varepsilon)$.

When Y is an arbitrary set, fix $y_\varepsilon = y$ in (9) we have

$$\varphi(z,y) < \inf_{B_r(x)\times Y} \varphi + \alpha\eta^2/4 \leq \inf_{x' \in B_r(x)} \varphi(x', y) + \alpha\eta^2/4. \tag{10}$$

Then we can apply the Smooth Variational Principle of Theorem 2.3 to the function $x' \to \varphi(x', y)$. The rest of the proof is similar. ■

3 Implicit Multifunction Theorems

Consider a multifunction between two Fréchet smooth spaces with a parameter p in a metric space. We derive coderivative infinitesimal conditions for the existence of an implicit multifunction and we also obtain a metric estimate for this implicit multifunction. We prove the general implicit multifunction theorem by reducing it to the following implicit multifunction theorem for $f(x,p) \le 0$. We will denote the implicit multifunction by G, i.e., $G(p) := \{x : f(x,p) \le 0\}$. This theorem builds upon an implicit function theorem established in [17] for proximal subdifferentials.

Theorem 3.1 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions, let (P,ρ) be a metric space and let U be an open set in $X \times P$. Suppose that $f : U \to \bar{R}$ satisfies the following conditions:*

(i) *there exists $(\bar{x},\bar{p}) \in U$ such that*

$$f(\bar{x},\bar{p}) \le 0;$$

(ii) *$p \to f(\bar{x},p)$ is upper semicontinuous at $\bar{p}$;*

(iii) *for any fixed p near $\bar{p}$, $x \to f(x,p)$ is lower semicontinuous;*

(iv) *there exists a $\sigma > 0$ such that, for any $(x,p) \in U$ with $f(x,p) > 0$, $\xi \in \partial_{F,x} f(x,p)$ implies that $\|\xi\| \ge \sigma$.*

Then there exist open sets $W \subset X$ and $V \subset P$ containing $\bar{x}$ and $\bar{p}$ respectively such that

(a) *for any $p \in V$, $W \cap G(p) \ne \emptyset$;*

(b) *for any $p \in V$ and $x \in W$,*

$$d(x, G(p)) \le \frac{f_+(x,p)}{\sigma},$$

where $f_+(x,p) := \max\{0, f(x,p)\}$;

(c) *if P is a Banach space with a Fréchet-smooth Lipschitz bump function then for any $(x,p) \in W \times V$, $x \in G(p)$,*

$$D^*G(p;x)(x^*) = \{p^* : (-x^*, p^*) \in \text{cone } \partial_F f_+(x,p)\}.$$

Proof. Let r' be a positive number such that $B_{r'}(\bar{x}) \times B_{r'}(\bar{p}) \subset U$ and let $r = r'/3$. Since $f(\bar{x}, p)$ is upper semincontinuous at $\bar{p}$ and $f(\bar{x}, \bar{p}) = 0$ there exists an open neighborhood V of $\bar{p}$ such that $V \subset B_{r'}(\bar{p})$ and $p \in V$ implies that $f(\bar{x}, p) < r\sigma$. We will show that V and $W := \text{int}B_r(\bar{x})$ satisfy the conclusion of the theorem. Let p be an arbitrary element of V. We show that $W \cap G(p) \neq \emptyset$. In fact, if this is not the case, then $f(x, p) > 0$ for any $x \in B_\tau(\bar{x}), \tau < r$. Choose τ close enough to r so that $f(\bar{x}, p) < \tau\sigma$. Invoking the decrease principle we have

$$0 \leq \inf_{x \in B_\tau(\bar{x})} f(x, p) \leq f(\bar{x}, p) - \tau\sigma < 0,$$

a contradiction.

To show the estimate (b), consider $x \in W$ and $p \in V$. If $B(x, f_+(x, p)/\sigma) \not\subset \text{int}B(\bar{x}, r')$ then $\|x - \bar{x}\| + f_+(x, p)/\sigma \geq r'$ or $f_+(x, p)/\sigma \geq 2r$. Since conclusion (a) implies that $d(x, G(p)) < 2r$ estimate (b) holds. Now we turn to the case when $B(x, f_+(x, p)/\sigma) \subset \text{int } B(\bar{x}, r')$. Take $\tau > f_+(x, p)/\sigma$ such that $B(x, \tau) \subset \text{int } B(\bar{x}, r')$. Since $f(x, p) < \tau\sigma$ an argument similar to the proof of (a) yields that there exists $z \in B(x, \tau)$ such that $f(z, p) \leq 0$. Thus, $d(x, G(p)) < \tau$. Letting $\tau \to f_+(x, p)/\sigma$ we arrive at estimate (b).

It remains to verify (c). Consider pair (x, p) with $x \in W \cap G(p)$ and let $p^* \in D^*G(p; x)(x^*)$. Then

$$(p^*, -x^*) \in N((p, x); \text{Graph } G) = \cup_{K>0} K\partial d((x, p), \text{Graph } G).$$

By definition there exists a C^1 function g with $g'(p, x) = (p^*, -x^*)$ and a positive constant K such that, for any $(q, y) \in P \times X$, we have

$$\begin{aligned} g(q, y) &\leq g(p, x) + Kd((q, y), \text{Graph } G) \\ &\leq g(p, x) + Kd(y, G(q)) \leq g(p, x) + (K/\sigma)f_+(y, q). \end{aligned}$$

Thus $(K/\sigma)f_+(y, q) - g(q, y)$ attains a minimum at (p, x), i.e., $(-x^*, p^*) \in (K/\sigma)\partial_F f_+(x, p)$. This establishes

$$D^*G(p; x)(x^*) \subset \{p^* : (-x^*, p^*) \in \text{cone } \partial_F f_+(x, p)\}.$$

The inverse inclusion follows directly from the inequality $\delta_{\text{Graph } G} \geq Kf_+$ for any $K > 0$. ■

Example 3.2 Let X and P be Banach spaces with Fréchet smooth equivalent norms. Let $F : X \times P \to Y$ be a single valued mapping with continuous (Fréchet) derivative $F'(x,p) = (F'_x(x,p), F'_p(x,p))$ and let $f(x,p) := \|F(x,p)\|$. Assume that f satisfies the conditions in Theorem 3.1. (It is easy to see that, for example,

$$\sigma B_Y \subset F'_x(x,p)B_X$$

is a sufficient condition for f to satisfy the infinitesimal subdifferential condition (iv)). Then an implicit function exists in a neighborhood of $\bar{p}$. We calculate the coderivative of G. Let $(-x^*, p^*) \in K\partial_F f(x,p)$. Then we have (by using directional derivatives)

$$\langle -x^*, e_x\rangle + \langle p^*, e_p\rangle \le \max_{\|\xi\|\le K} \langle \xi, F'_x e_x + F'_p e_p\rangle \ \forall e_x \in X, e_p \in P.$$

It follows that

$$\min_{\|e_x\|\le 1, \|e_p\|\le 1} \max_{\|\xi\|\le K} \langle (F'_x)^*\xi - x^*, e_x\rangle + \langle (F'_p)^*\xi + p^*, e_p\rangle \ge 0.$$

Therefore, there exists $\xi \in KB_{Y^*}$ such that

$$x^* = (F'_x)^*\xi \text{ and } p^* = -(F'_p)^*\xi.$$

By conclusion (c) in Theorem 3.1 we have

$$D^*G(p;x)(x^*) = \{-(F'_p)^*\xi : x^* = (F'_x)^*\xi, \xi \in Y^*\}.$$

In particular, when F'_x is invertible,

$$D^*G(p;x)(x^*) = \{-((F'_x)^{-1}F'_p)^*x^*\}.$$

■

We now turn to the implicit multifunction

$$G(p) := \{x : 0 \in F(x,p)\}$$

where $F : X \times P \to 2^Y$ is a multifunction. Define

$$f(x,p) := d(0, F(x,p)).$$

Then $G(p) = \{x : f(x,p) \le 0\}$. Using this relation we convert the problem of finding the implicit multifunction function for $0 \in F(x,p)$ to one for

$f(x,p) \leq 0$. The key is the relationship between the infinitesimal regularity coderivative condition for a multifunction F and that of the subdifferential condition for f. To illustrate the idea we start with the case when the value of F is locally compact and the Banach space Y admits Fréchet-smooth norms. We denote the projection of a vector v onto a closed set S by $\mathrm{pr}(v,S) := \{s \in S : \|v - s\| = d(v,S)\}$.

Lemma 3.3 *Let X be a Banach space with Fréchet-smooth Lipschitz bump functions, let Y be a Banach space with a Fréchet-smooth norm, let $U \subset X$ be an open set and let $F : U \to 2^Y$ be a locally compact-valued upper semicontinuous multifunction. Denote $f(x) := d(0, F(x))$. Suppose*

(i) *for any $x \in U$ with $0 \notin F(x)$*

$$\sigma \leq \inf\{\|x^*\| : y \in \mathrm{pr}(0, F(x)), x^* \in D^*F(x;y)(y^*), \|y^*\| = 1\}.$$

Then

(ii) *for any $x \in U$ with $f(x) > 0$, $\xi \in \partial_F f(x)$ implies that $\|\xi\| \geq \sigma$.*

Proof. It is easy to see that when F is upper semicontinuous f is lower semicontinuous. Let $\xi \in \partial_F f(x)$ where $f(x) > 0$. By the definition of the subdifferential there is a Fréchet smooth function g such that $g'(x) = \xi$ and $f - g$ attains a local minimum 0 at x. Since F is locally compact-valued, $\mathrm{pr}(0, F(x)) \neq \emptyset$. Let $y \in \mathrm{pr}(0, F(x))$. Then $\|y\| = f(x)$ and we have, for x' sufficiently close to $\bar{x}$,

$$\begin{aligned} & f(x) - g(x) = \|y\| - g(x) = \|y\| + \delta_{\mathrm{Graph}\ F}(x,y) - g(x) \\ \leq \quad & f(x') - g(x') \leq \|y'\| + \delta_{\mathrm{Graph}\ F}(x',y') - g(x'). \end{aligned}$$

That is to say

$$(x',y') \to \|y'\| + \delta_{\mathrm{Graph}\ F}(x',y') - g(x')$$

attains a local minimum at (x,y). Note that $\|y\| > 0$ and, therefore, $(x',y') \to g(x') - \|y'\|$ is differentiable at (x,y). Thus,

$$\xi = g'(x) \in D^*F(x,y)(\|y\|').$$

Observing that $\|y\|'$ is a unit vector we obtain

$$\|\xi\| \geq \sigma.$$

■

When F is not locally compact-valued, $\mathrm{pr}(0, F(x))$ may be empty for all x and Lemma 3.3 may fail, as is shown by the following example.

Example 3.4 Let $X := l_2$ and let e_n be the standard basis for X. Define $S := \{e_n + e_1/n : n = 1, 2, ...\}$ and $F(x) := \|x\|S$. Then it is easy to check that $d(0, F(x)) = \|x\|$ is not attained for any $x \neq 0$.

Fortunately we can establish a similar relationship by using 'approximate projections'. For $\eta > 0$, we denote the η-approximate projection of v to S by $\mathrm{pr}_\eta(v, S) := \{s \in S : \|s - v\| \leq d(v, S) + \eta\}$. The idea is to view f as a marginal function.

$$f(x) = d(0, F(x)) = \inf\{\|y\| + \delta_{\text{Graph F}}(x, y)\}.$$

Lemma 3.5 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions, let $U \subset X$ be an open set and let $F : U \to 2^Y$ be a close-valued upper semicontinuous multifunction. Let $f(x) := d(0, F(x))$. Suppose*

(i) *for any $x \in U$ with $0 \notin F(x)$*

$$\sigma \leq \liminf_{\eta \to 0} \{\|x^*\| \quad : \quad x^* \in D^*F(x'; y')(y^*), \|y^*\| = 1 \quad \text{with } x' \in B_\eta(x), y' \in \mathrm{pr}_\eta(0, F(x'))\}.$$

Then

(ii) *for any $\bar{x} \in U$ with $f(\bar{x}) > 0$, $\xi \in \partial_F f(\bar{x})$ implies that $\|\xi\| \geq \sigma$.*

Proof. Consider

$$\varphi(x', y') = \|y'\| + \delta_{\text{Graph } F}(x', y').$$

Since F is a closed-valued upper semicontinuous multifunction the graph of F is a closed set. Thus, φ is lower semicontinuous. Moreover, $f(x') = d(0, F(x')) = \inf_{y' \in Y} \varphi(x', y')$. Let $\xi \in \partial_F f(x)$ where $f(x) > 0$. Then f is lower semicontinuous at x. (In fact, when F is upper semicontinuous it is not hard to verify directly that f is lower semicontinuous.) Take η small enough so that $\|x' - x\| < \eta$ implies that $f(x') \geq f(x)/2 > 0$. Applying Theorem 2.7 with $\varepsilon = \eta$ we obtain that there exist (u_η, v_η) and $(\xi_\eta, \zeta_\eta) \in \partial_F \varphi(u_\eta, v_\eta)$ such that $\|x - u_\eta\| < \eta$,

$$0 < f(u_\eta) < \varphi(u_\eta, v_\eta) < f(u_\eta) + \eta \tag{11}$$

and

$$\|\xi_\eta - \xi\| < \eta, \quad \|\zeta_\eta\| < \eta. \tag{12}$$

It follows from (11) that $v_\eta \in \mathrm{pr}_\eta(0, F(u_\eta))$. By the sum rule of Theorem 2.4 there exist (x_η, y_η), (x'_η, y'_η) close to (u_η, v_η) and a subgradient y^*_η of the norm function $\|y\|$ at the point y'_η such that $y_\eta \in \mathrm{pr}_\eta(0, F(x_\eta))$, $\|y'_\eta\| > 0$ and

$$(\xi_\eta, \zeta_\eta) \in (0, y^*_\eta) + N((x_\eta, y_\eta), \mathrm{Graph}\ F) + \eta(B_{X^*} \times B_{Y^*}),$$

i.e., there exists $(\xi'_\eta, \zeta'_\eta) \in \eta(B_{X^*} \times B_{Y^*})$ such that

$$\xi_\eta - \xi'_\eta \in D^*F(x_\eta, y_\eta)(y^*_\eta - \zeta_\eta + \zeta'_\eta). \tag{13}$$

Rewriting (13) as

$$(\xi_\eta - \xi'_\eta)/\|y^*_\eta - \zeta_\eta + \zeta'_\eta\| \in D^*F(x_\eta, y_\eta)((y^*_\eta - \zeta_\eta + \zeta'_\eta)/\|y^*_\eta - \zeta_\eta + \zeta'_\eta\|) \tag{14}$$

and noting that $\|y^*_\eta - \zeta_\eta + \zeta'_\eta\| \geq 1 - 2\eta$, it follows from assumption (i) that

$$\liminf_{\eta \to 0} \|\xi_\eta\| = \liminf_{\eta \to 0} \|\xi_\eta - \xi'_\eta\| / \|y^*_\eta - \zeta_\eta + \zeta'_\eta\| \geq \sigma.$$

Relation (12) then implies that

$$\|\xi\| \geq \sigma.$$

■

Combining Theorem 3.1, Lemma 3.3 and Lemma 3.5 we have the following implicit multifunction theorem.

Theorem 3.6 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions, let (P, ρ) be a metric space, and let U be an open set in $X \times P$. Suppose that $F : U \to 2^Y$ is a closed-valued multifunction satisfying:*

(i) *there exists $(\bar{x}, \bar{p}) \in U$ such that*

$$0 \in F(\bar{x}, \bar{p}),$$

(ii) *$p \to F(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$,*

(iii) *for any fixed p near $\bar{p}$, $x \to F(x, p)$ is upper semicontinuous, and*

(iv) *there exists $\sigma > 0$ such that for any $(x, p) \in U$ with $0 \notin F(x, p)$*

$$\sigma \leq \liminf_{\eta \to 0} \{\|x^*\| \ : \ x^* \in D^*F(x', p; y')(y^*), \|y^*\| = 1 \text{ with } x' \in B_\eta(x), y' \in \mathrm{pr}_\eta(0, F(x', p))\}.$$

Then there exist open sets $W \subset X$ and $V \subset P$ containing $\bar{x}$ and $\bar{p}$ respectively such that

(a) *for any $p \in V$, $W \cap G(p) \neq \emptyset$,*

(b) *for any $p \in V$ and $x \in W$, $d(x, G(p)) \leq d(0, F(x,p))/\sigma$, and*

(c) *if P is a Banach space with a Fréchet-smooth Lipschitz bump function then for any $(x,p) \in W \times V$, $x \in G(p)$,*

$$D^*G(p;x)(x^*) = \{p^* : (-x^*, p^*) \in \text{cone } \partial_F d(0, F(x,p))\}.$$

If, in addition, $F : U \to 2^Y$ is a locally compact-valued multifunction and Y admits a Fréchet-smooth norm, then we can replace condition (iv) *by the following simpler condition*

(iv') *there exists $\sigma > 0$ such that for any $(x,p) \in U$ with $0 \notin F(x,p)$*

$$\sigma \leq \inf\{\|x^*\| : y \in \text{pr}(0, F(x,p)), x^* \in D^*F(x,p;y)(y^*), \|y^*\| = 1\}.$$

In Theorem 3.6 the characterization of the coderivative for the implicit multifunction is given in terms of the Fréchet subdifferential of the scalar function $d(0, F(x,p))$. A natural question arises: How does the coderivative of the implicit multifunction G relate to the coderivative of F? A partial answer is contained in the following proposition. We will use the sequential limiting subdifferential and normal cone defined in [42]. Let us recall their definitions first. Let $f : X \to \bar{R}$ be a lower semicontinuous function and let S be a closed subset of X. The sequential limiting subdifferential of f at x is defined by

$$\tilde{\partial} f(x) := \{w^* - \lim_{i\to\infty} v_i : v_i \in \partial_F f(x_i),\ (x_i, f(x_i)) \to (x, f(x))\},$$

and the sequential limiting normal cone of S at $x \in S$ is defined by

$$\tilde{N}(S,x) := \{w^* - \lim_{i\to\infty} v_i : v_i \in N(S, x_i),\ S \ni x_i \to x\}$$

Proposition 3.7 *Under the assumptions of Theorem 3.6, the following are true:*
(a) *for any $x^* \in X^*$,*

$$\cup_{y^* \in Y^*}\{p^* : (-x^*, p^*) \in D^*F(x,p;0)(y^*)\} \subset D^*G(p;x)(x^*); \tag{15}$$

(b) *for any* x^* *and*

$$p^* \in D^*G(p;x)(x^*) \tag{16}$$

and any $\varepsilon > 0$ *there exists* $(x_\varepsilon, p_\varepsilon, y_\varepsilon) \in \text{Graph } F$ *and* $(x_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*)$ *such that* $\|x - x_\varepsilon\| < \varepsilon$, $\|p - p_\varepsilon\| < \varepsilon$, $\|y_\varepsilon\| < \varepsilon$,

$$(-x_\varepsilon^*, p_\varepsilon^*) \in D^*F(x_\varepsilon, p_\varepsilon; y_\varepsilon)(y_\varepsilon^*) \tag{17}$$

and

$$\|x^* - x_\varepsilon^*\| < \varepsilon, \|p^* - p_\varepsilon^*\| < \varepsilon, \tag{18}$$

and
(c) *If, in particular, the multifunction* F *is regular at point* $(x, p, 0) \in \text{Graph } F$ *which means*

$$N((x,p,0), \text{Graph } F) = \tilde{N}((x,p,0), \text{Graph } F),$$

then we have equality in (15).

Proof. (a) Let p^* belong to the left-hand side of the inclusion (15). Then we have that for some $y^* \in Y^*$,

$$(-x^*, p^*, y^*) \in N((x,p,0), \text{Graph } F).$$

By definition there exists a C^1 function g such that $g'(x,p,0) = (-x^*, p^*, y^*)$ and $\delta_{\text{Graph } F} - g$ attains a minimum at $(x,p,0)$. Then

$$\delta_{\text{Graph } G}(q,y) - g(y,q,0) = \delta_{\text{Graph } F}(y,q,0) - g(y,q,0)$$

attains a minimum at (x,p). Thus,

$$p^* \in D^*G(p;x)(x^*).$$

To show (b) and (c) let $p^* \in D^*G(p;x)(x^*)$. Then

$$(p^*, -x^*) \in N((p,x); \text{Graph } G) = \cup_{K>0} K\partial d((p,x), \text{Graph } G).$$

By definition there exists a C^1 function g such that $g'(p,x) = (p^*, -x^*)$ and a positive constant K such that

$$\begin{aligned} g(p,x) &\leq g(q,y) + Kd((q,y), \text{Graph } G) \\ &\leq g(q,y) + Kd(y, G(q)) \leq g(q,y) + (K/\sigma)d(0, F(y,q)). \end{aligned} \tag{19}$$

Observing that $d(0, F(y,q)) = \inf_{u\in Y}\{\|u\| + \delta_{\text{Graph }F}(y,q,u)\}$ it follows from inequality (19) that

$$(p^*, -x^*) \in \partial \inf_{u\in Y}\{(K/\sigma)\|u\| + \delta_{\text{Graph }F}(y,q,u)\}.$$

Then (b) follows from Theorem 2.7. Moreover, if F is regular, then (c) follows from (17) by taking limits. ■

An immediate corollary of Theorem 3.6 is the follwoing psuedo-Lipschitz property of the implicit multifunction which is often refered to as Aubin continuity.

Corollary 3.8 *Let the conditions in Theorem 3.6 be satisfied. In addition assume that $F : U \to 2^Y$ satisfies*

(ii') *$(x,p) \to F(x,p)$ is partially pseudo-Lipschitz in p with rank L around $(\bar{x}, \bar{p}, 0)$, i.e., there exist open neighborhoods O of 0, such that, for any $(x,p_1), (x,p_2) \in U$,*

$$O \cap F(x,p_2) \subset F(x,p_1) + L\rho(p_1,p_2)B_Y$$

Then $G(p)$ is pseudo-Lipschitz with rank L/σ, i.e., there exist open sets $W \subset X$ and $V \subset P$ containing $\bar{x}$ and $\bar{p}$ respectively such that

$$W \cap G(p_2) \subset G(p_1) + \frac{L}{\sigma}\rho(p_1,p_2)B_X \quad \forall p_1, p_2 \in V.$$

Proof. Let W, V be as in the conclusion of Theorem 3.6 and let $p_1, p_2 \in V$. Consider an arbitrary element $x \in W \cap G(p_2)$. Then $0 \in F(x,p_2)$ and therefore $0 \in O \cap F(x,p_2)$. Conclusion (b) of Theorem 3.6 implies that $d(x, G(p_1)) \le d(0, F(x,p_1))/\sigma$. Since $0 \in O \cap F(x,p_2) \subset F(x,p_1) + L\rho(p_1,p_2)B_Y$ we have

$$d(x, G(p_1)) \le d(0, F(x,p_1))/\sigma \le L\rho(p_1,p_2)/\sigma,$$

i.e., $x \in G(p_1) + \frac{L}{\sigma}\rho(p_1,p_2)B_X$. Since $x \in W \cap G(p_2)$ is arbitrary we have

$$W \cap G(p_2) \subset G(p_1) + \frac{L}{\sigma}\rho(p_1,p_2)B_X,$$

as was to be shown. ■

Remark 3.9 An obvious sufficient condition for (iv) in Theorem 3.6 is

(iv") there exists a $\sigma > 0$ such that for any $(x,p) \in U$ with $0 \notin F(x,p)$, $x^* \in F(x,p;y)(y^*)$ implies that $\|x^*\| \geq \sigma \|y^*\|$.

While condition (iv") is easier to state than conditions (iv) and (iv') in Theorem 3.6 the latters are much weaker and easier to use. We illustrate this with the following example.

Example 3.10 *Define a multifunction H by*

$$\text{graph H} := \{(x,y) \in R^2 : y = x, y = 2x\} \cup \bigcup_{n=1}^{\infty} \{(x, 1/n) \in R^2 : x \in [1/2n, 1/n]\}$$

and set $F(x,p) = H(x) + p$. Then F satisfies (iv') *in Theorem 3.6 but not* (iv")*.*

Remark 3.11 When X and Y are Hilbert spaces and F is a closed and convex valued multifunction that is Lipschitz in x, Dien and Yen [22] established the following sufficient infinitesimal condition for the existence of an implicit multifunction to $0 \in F(x,p)$: there exist $r, s > 0$ such that for all (x,p) sufficiently close to $(\bar{x}, \bar{p})$, for all $\|y^*\| = 1$ with $s(y^*, F(x,p)) < \infty$, and all $x^* \in \partial_{C,x} s(y^*, F(x,p))$ there exists a unit vector $u \in X$ satisfying

$$s(y^*, F(x,p)) + r\langle x^*, u\rangle \geq s. \tag{20}$$

Here $s(\cdot, F(x,p))$ is the support function for $F(x,p)$ and $\partial_{C,x}$ signifies the Clarke generalized gradient with respect to x. Observe that if $0 \notin F(x,p)$ and $x^* \in \partial_{F,x} d(0, F(x,p))$ then there exists a unique unit vector y^* with $s(y^*, F(x,p)) < \infty$ such that $d(0, F(x,p)) = -s(y^*, F(x,p))$. Then

$$x^* \in \partial_{F,x} d(0, F(x,p)) \subset \partial_{C,x} d(0, F(x,p)) \subset -\partial_{C,x} s(y^*, F(x,p)).$$

Therefore,

$$\|x^*\| \geq \langle x^*, u\rangle \geq s/r + d(0, F(x,p)) \geq s/r.$$

Thus condition (20) implies the infinitesimal condition (iv) in Theorem 3.1 with $f(x,p) = d(0, F(x,p))$.

4 Open Covering, Metric Regularity and Aubin Continuity for Multifunctions

The classical implicit function theorem is closely related to open mapping theorems. What can we expect from the implicit multifunction theorem? It turns out that the implicit multifunction theorem is also a very potent result. In this section, we deduce directly from the implicit multifunction theorem the open covering theorem and coderivative sufficient conditions for both the metric regularity and Aubin continuity for multifunctions. It is well known that the open covering theorem is closely related to metric regularity and the pseudo-Lipschitzian property [5, 43, 48]. In fact, it is shown by Mordukhovich [43] and Mordukhovich and Shao [48] that, when appropriately defined, they are equivalent properties for multifunctions in more general setting of Asplund spaces. Their papers also contain coderivative criteria for these properties with exact formulae of constants in terms of coderivatives. Thus, in principle we need only to deduce one of these results from the implicit multifunction theorem. However, we choose to give all the three proof because they are very short. It illustrates the power of the metric estimate in the implicit multifunction theorem. We also weaken somewhat these sufficient conditions. In what follows, F^{-1} represents the preimage of the multifunction F defined by $F^{-1}(y) := \{x : y \in F(x)\}$. Recall that a multifunction $F : X \to 2^Y$ is metrically regular at $(\bar{x}, \bar{y}) \in \text{Graph } F$ provided that there exist a constant $r > 0$ and neighborhoods W and V of $\bar{x}$ and $\bar{y}$ respectively such that, for any $x \in W$ and $y \in V$,

$$d(x, F^{-1}(y)) \leq rd(y, F(x)).$$

Now we can precisely state our deduction.

Theorem 4.1 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions. Let U be an open set in $X \times Y$ and let $F : U \to 2^Y$ be a closed-valued multifunction satisfying:*

(i) *there exists $(\bar{x}, \bar{y}) \in U$ such that*

$$\bar{y} \in F(\bar{x}),$$

(ii) *F is upper semicontinuous,*

(iii) *there exists* $\sigma > 0$ *such that for any* $(x, y) \in U$, $y \notin F(x)$

$$\sigma \leq \liminf_{\eta \to 0} \{\|x^*\| \;\; : \;\; x^* \in D^*F(x'; y')(y^*), \|y^*\| = 1, \\ \text{with } x' \in B_\eta(x), y' \in \text{pr}_\eta(y, F(x'))\}.$$

Then

(a) (Open Covering) *there exists an open set* W *containing* $\bar{x}$ *such that, for any* $B_r(\bar{x}) \subset W$,

$$\text{int } B_{\sigma r}(\bar{y}) \subset F(B_r(\bar{x}));$$

(b) (Metric Regularity) *there exist neighborhoods* W *and* V *of* $\bar{x}$ *and* $\bar{y}$ *respectively such that, for any* $x \in W$ *and* $y \in V$,

$$d(x, F^{-1}(y)) \leq \frac{1}{\sigma} d(y, F(x));$$

(c) (Aubin Continuity) F^{-1} *is Aubin continuous at* $(\bar{y}, \bar{x})$.

Proof. (a) Let $P = Y$ and apply Theorem 3.6 to $F(x) - y$. Let W and V be as in the conclusion of Theorem 3.6. Taking a smaller W if necessary, we may assume that $B_r(\bar{x}) \subset W$ implies $B_{\sigma r}(\bar{y}) \subset V$. Note that $G(y) = F^{-1}(y)$ and $d(0, F(x) - y) = d(y, F(x))$ is Lipschitz in y with rank 1. For any $\|y - \bar{y}\| < \sigma r$, we have $y \in V$ and, therefore,

$$d(\bar{x}, F^{-1}(y)) \leq \frac{d(y, F(\bar{x}))}{\sigma} \leq \frac{d(y, F(\bar{x})) - d(\bar{y}, F(\bar{x}))}{\sigma} < r.$$

That is to say int $B_{\sigma r}(\bar{y}) \subset F(B_r(\bar{x}))$.

(b) Let $P = Y$ and apply Theorem 3.6 to $F(x) - y$. Observing that $G(y) = F^{-1}(y)$ and $d(0, F(x) - y) = d(y, F(x))$, the conclusion follows directly.

(c) Let $P = Y$ and apply Corollary 3.8 to $F(x) - y$. ■

Remark 4.2 (a) When $F(x)$ is locally compact-valued we can replace condition (iii) in Theorem 4.1 by

(iii') there exists $\sigma > 0$ such that, for any $(x, y) \in U$, $y \notin F(x)$,

$$\sigma \leq \inf\{\|x^*\| : x^* \in D^*F(x; y')(y^*), \|y^*\| = 1, \text{ with } y' \in \text{pr}(y, F(x))\}.$$

(b) It is obvious that the following condition

(iii") there exists a $\sigma > 0$ such that, for any $(x, y) \in U$, $y \in F(x)$,

$$\sigma \leq \inf\{\|x^*\| \quad : \quad x^* \in D^*F(x; y)(y^*), \|y^*\| = 1\}.$$

implies condition (iii) in Theorem 4.1. However, (iii") is stronger than (iii). In fact, consider the multifunction H defined in Example 3.2. H evidently is an open covering at a linear rate. It is easy to verify that condition (iii) holds for H while condition (iii') does not.

Remark 4.3 In conclusion (a) of the above theorem we actually get more than the open mapping property. This property is referred to as "open covering at a linear rate" [43, 47] or "fat" open covering [63]. Other discussions on the open covering property can be found in [2, 9, 26, 27, 28, 32].

Moreover, it follows from the proof of Theorem 3.1 that we can obtain the conclusion

$$\text{int } B_{\sigma r}(y) \subset F(B_r(x)) \quad \forall (x, y) \in \text{Graph } F \cap U$$

in conclusion (a) of Theorem 4.1 provided that $B_{3r}(x) \times B_{\sigma r}(y) \subset U$. This essentially means that the above inclusion holds uniformly for all $(x, y) \in U$ having the distance $\max(3r, \sigma r)$ from the boundary of U. This observation is useful in what follows.

When there exists a positive lower bound for the σ in the infinitesimal condition in Theorem 4.1 for all x equidistant from $\bar{x}$, we show that the local covering result in Theorem 4.1 implies a global covering result. Similar global covering results were discussed by Ioffe [31] and Warga [64]. Our proof follows the scheme suggested by Ioffe [31] and exploits his idea of passing from a local surjectivity condition to a global one (see [32]).

Theorem 4.4 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions. Let $F : X \to 2^Y$ be a closed-valued upper semicontinuous multifunction with $\bar{y} \in F(\bar{x})$. Assume that there exists a lower semicontinuous function $\sigma : [0, +\infty) \to (0, +\infty)$ such that the multifunction F satisfies the following global infinitesimal regularity condition*

$$\sigma(t) \leq \liminf_{\eta \to 0} \inf_{\|x - \bar{x}\| = t, y \notin F(x)} \{\|x^*\| : x^* \in D^*F(x', y')(y^*), \|y^*\| = 1,$$
$$\|x' - x\| < \eta \text{ and } y' \in \text{pr}_\eta(y, F(x'))\}.$$

Then, for any $a \in [0, a_\infty)$,

$$\text{int } B(\bar{y}, \int_0^a \sigma(t)dt) \subset F(B(\bar{x}, a))$$

where $a_\infty := \sup\{a : \int_0^a \sigma(t)dt < +\infty\}$.

Proof. Without loss of generality we may assume that $(\bar{x}, \bar{y}) = (0, 0)$. For any $K > 0$, define

$$\sigma_K(t) = \inf\{\sigma(t') + K|t - t'| : t' \in [0, +\infty)\}.$$

Then σ_K is Lipschitz for all K and $\sigma_K \uparrow \sigma$ pointwise as $K \to +\infty$. We prove that, for any $K > 0$ and any $\theta \in (0, 1)$,

$$B(0, \int_0^a \theta\sigma_K(t)dt) \subset F(B(0, a)) \quad \forall a \in [0, a_\infty). \tag{21}$$

This clearly implies the conclusion of the theorem because, for any $y \in \text{int } B(0, \int_0^a \sigma(t)dt)$, there exist $K > 0$ and $\theta \in (0, 1)$ such that $y \in B(0, \int_0^a \theta\sigma_K(t)dt)$.

Fix $K > 0$ and $\theta \in (0, 1)$. For any $y \in B_Y$ we define

$$\bar{a} := \sup\{a \in [0, a_\infty) : \int_0^a \theta\sigma_K(t)dt \cdot y \in F(B(0, a))\}.$$

To prove (21) we need only show that $\bar{a} = a_\infty$. Suppose, to the contrary that $\bar{a} < a_\infty$. Choose $b \in (\bar{a}, a_\infty)$. Then there exists $\eta \in (0, b - \bar{a})$ such that for any $t, t' \in [0, b]$ and $|t' - t| < \eta$,

$$\sigma(t') > \theta\sigma_K(t) \tag{22}$$

and

$$\sigma := \inf\{\sigma(t) : t \in [0, b]\} > 0. \tag{23}$$

Let a^n and a_n be sequences such that $a^n \downarrow \bar{a}$ and $a_n \uparrow \bar{a}$ and define $y^n := \int_0^{a^n} \theta\sigma_K(t)dt \cdot y$ and $z_n := \int_0^{a_n} \theta\sigma_K(t)dt \cdot y$. Then $y^n \notin F(B(0, a^n))$ and $z_n \in F(B(0, a_n))$. Choose $x_n \in B(0, a_n)$ such that $z_n \in F(x_n)$. We show that $\|x_n\| \to \bar{a}$. In fact, if this is not true then there exists $r < \eta/3$ such that $\|x_n\| < \bar{a} - r$. Take n sufficiently large so that $\|y^n - z_n\| < \sigma r$. Then by Theorem 4.1 and Remark 4.3

$$y^n \in \text{int } B(z_n, \sigma r) \subset F(B(x_n, r)) \subset F(B(0, \bar{a}))$$

a contradiction.

Next we define $y_n := \int_0^{\|x_n\|} \theta\sigma_K(t)dt \cdot y$. Then

$$\left\| \frac{y^n - y_n}{a^n - \|x_n\|} \right\| = \frac{1}{a^n - \|x_n\|} \int_{\|x_n\|}^{a^n} \theta\sigma_K(t)dt \to \theta\sigma_K(\bar{a}).$$

By (22) for $r < \eta/6$ and n sufficiently large, $\|x - x_n\| < r$ implies that

$$\sigma(\|x\|) > \theta\sigma_K(\bar{a}). \tag{24}$$

Choose n large enough so that (24) holds along with $a^n - \|x_n\| < r$ and

$$\|y^n - y_n\| \leq \sigma(\|x_n\|)(a^n - \|x_n\|).$$

Then, using Theorem 4.1 and Remark 4.3 again, we have

$$y^n \in \text{int } B(y_n, \sigma(\|x_n\|)(a^n - \|x_n\|)) \subset F(B(x_n, a^n - \|x_n\|)) \subset F(B(0, a^n)),$$

a contradiction. ■

Remark 4.5 Let $h : X \to Y$ be a C^1 function and let S be a closed subset of X. Define

$$F(x) = \begin{cases} h(x) & x \in S, \\ \emptyset & x \notin S. \end{cases} \tag{25}$$

Then F is metrically regular at $(\bar{x}, h(\bar{x}))$ if and only if h is metrically regular with respect to S at $\bar{x}$ (see [9] for the definition). In [9, Theorem 4.3] it was shown that h is regular with repect to S at $\bar{x}$ provided that there exists a $\sigma > 0$ such that, for any $x \in S$ close enough to $\bar{x}$, any unit vector $y^* \in Y^*$ and any $x^* \in N(x; S)$,

$$\|(h'(x))^* y^* + x^*\| \geq \sigma. \tag{26}$$

(In [9] h is required to be strictly differentiable at $\bar{x}$, a condition slightly weaker than C^1.) We will show that [9, Theorem 4.3] follows from (b) of Theorem 4.1. The sufficient conditions for metric regularity in (b) of Theorem 4.1 and [9, Theorem 4.3] are, in essence, conditions in terms of the normal cone to the graph of the multifunction F. Note that many sufficient conditions for metric regularity in terms of the tangent cone associated with the graph of F can be deduced from [9, Theorem 4.3] (and, therefore, Theorem 4.1) as demonstrated in [9]. We now turn to proving that the conditions in [9, Theorem 4.3] imply the conditions in Theorem 4.1. Let $\bar{y} = h(\bar{x})$. The F defined in (25) satisfies (i) and (ii) in Theorem 4.1. It remains to show that (26) implies the infinitesimal condition (iii) in Theorem 4.1. Consider (x, y) close to $(\bar{x}, \bar{y})$, an unit vector $y^* \in Y^*$ and a coderivative $z^* \in D^*F(x, y)(y^*)$. Then there exists a C^1 function g with $g'(x, y) = (z^*, -y^*)$ such that, for all $(x', y') \in \text{Graph } F$ close to (x, y),

$$g(x', y') \leq 0 = g(x, y).$$

Then, for $x' \in S$ close to $\bar{x}$,

$$f(x') := g(x', h(x')) \leq 0 = f(x).$$

Thus,

$$x^* = f'(x) = z^* - (h'(x))^* y^* \in N(x; S).$$

It follows from (26) that $\|z^*\| = \|(h'(x))^* y^* + x^*\| \geq \sigma$. ■

5 Stability of Implicit Multifunctions and Infinitesimal Conditions for Solvability

We now take a different view of the implicit multifunction G given by Theorem 3.1. Recall that

$$G(p) := \{x : f(x, p) \leq 0\}, \tag{27}$$

where $f(x, p); = d(0, F(x, p))$. The condition (iv) in Theorem 3.1 ensures the nonemptiness of $G(p)$ on V, i.e., the solvability of the inclusion

$$0 \in F(x, p) \tag{28}$$

for all $p \in V$. Moreover, we also have the metric estimate

$$d(x, G(p)) \leq \frac{d(0, F(x, p))}{\sigma}, \ \forall x \in W. \tag{29}$$

Let us consider a parameter $\alpha > 0$ and the corresponding implicit multifunction under α-perturbations:

$$G_\alpha(p) := \{x : 0 \in F(x, p) + \alpha B_Y\} = \{x; f(x, p) \leq \alpha\}. \tag{30}$$

It is obvious that

$$G(p) \subset G_\alpha(p), \ \forall \alpha > 0. \tag{31}$$

Interestingly, it follows from Theorem 3.1 that the opposite inclusion holds with an appropriate tolerance. More precisely, we have the following corollary.

Corollary 5.1 *Let assumption of Theorem 3.1 hold. Then, for $\alpha > 0$ small enough,*

$$d(x, G_\alpha(p)) \leq \frac{d(0, F(x,p) + \alpha B_Y)}{\sigma}, \ \forall x \in W. \tag{32}$$

In particular,

$$W \cap G_\alpha(p) \subset G_{\alpha'}(p) + \frac{\alpha - \alpha'}{\sigma} B_X. \tag{33}$$

Proof. Straightforward.

Remark 5.2 Note that relation (32) is closely related to Robinson's stability results for systems of equations and inequalities. The term "stability" is used to emphasize the fact that (32) implies that the solution of the inclusion

$$0 \in F(x,p) + \alpha B_Y \tag{34}$$

is close to the solution of (28). Moreover, the distance to that solution is less than α/σ, where α measures the perturbation of the right hand side of (29).

The next proposition establishes that the "stability condition" (32) implies the infinitesimal solvability condition (iv) in Theorem 3.1.

Proposition 5.3 *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions, let (P, ρ) be a metric space and let U be an open set in $X \times P$. Suppose that $F : U \to 2^Y$ is a closed-valued multifunction satisfying conditions* (i), (ii) *and* (iii) *in Theorem 3.6. Then the following two conditions are equivalent:*

(a) *There exist $\sigma > 0$ and a neighborhood U of (x, p) such that, for any $(x,p) \in U$ with $f(x,p) > 0$, $\xi \in \partial_F f(x,p)$ implies that $\|\xi\| \geq \sigma$.*

(b) *There exist neighborhoods V and W of $\bar{p}$ and $\bar{x}$ respectively such that for any $p \in V$, $x \in W$, and $\alpha > 0$,*

$$d(x, G_\alpha(p)) \leq \frac{d(0, F(x,p) + \alpha B_Y)}{\sigma}.$$

Proof. We need only prove that (b) implies (a). Let U be a neighborhood of $(\bar{x}, \bar{p})$ such that $U \subset W \times V$. Consider $(x, p) \in U$ with $f(x, p) > 0$ and $\xi \in Df(x, p)$. Then there exists a C^1 function g such that $g'(x) = \xi$ and $f - g$ attains a minimum at x. Let us choose $t > 0$ and set $\alpha = f(x, p) - t\sigma$. Then it follows from (b) that there exists $x_t \in G_\alpha(p)$ such that

$$\|x_t - x\| \leq \frac{f(x, p) - \alpha}{\sigma} + t^2 = t + t^2.$$

Define $e_t = (x_t - x)/t$. Then $x_t = x + te_t$ and $\|e_t\| \leq 1 + t$. Since $x_t \in G_\alpha(p)$ we have $f(x_t, p) \leq \alpha = f(x, t) - t\sigma$. Thus

$$\begin{aligned} -\sigma &\geq \frac{f(x_t, p) - f(x, p)}{t} \geq \frac{g(x_t, p) - g(x, p)}{t} \\ &= \langle \nabla g(x + \theta_t t e_t), e_t \rangle \geq -\|\nabla g(x + \theta_t t e_t)\| \cdot \|e_t\| \\ &\geq -\|\nabla g(x + \theta_t t e_t)\|(1 + t), \end{aligned}$$

where $\theta_t \in (0, 1)$. Taking limits as $t \to 0$ yields $\|\xi\| \geq \sigma$, which completes the proof. ■

6 The Robinson-Ursescu Open Mapping Theorem and a Refinement of the Multidirectional Mean Value Inequality

The Robinson-Ursescu [56, 62] open mapping theorem for a closed convex multifunction encompasses many results in classical linear functional analysis as special cases. Various applications of this open mapping theorem can be found in [10, 56, 57]. In this section we show that method we used in the previous sections can also be used to give an alternative proof for the Robinson-Ursescu open mapping theorem [56, 62] for convex multifunctions. Interestingly, to do so, we need a refined form of the multidirectional mean valued inequality. We hope this will shed some light on the relationship between the Robinson-Ursescu open mapping theorem and various results we discussed in the previous sections. The refined multidirectional mean valued inequality is a result of independent ineterest. It gives more precise information on the interpolating point while requiring stronger conditions on the underline function.

6.1 A refined multidirectional mean value inequality

In the multidirectional mean value inequality of Theorem 2.4 it is only asserted that the point z is close to $[x, Y]$. We now show that when f is a locally Lipschitz quasi-differentiable function, the multidirectional mean value inequality can be refined to ensure that $z \in [x, Y]$. In particular, when f is a Gateaux differentiable function this reduces to [16, Theorem 4.1]. This refinement is what we need to derive the Robinson-Ursescu open mapping theorem. It is also of independent interest. We start by recalling Pshenichnyĭ's definition of quasi-differentiable function [54].

Definition 6.1 *A function f on X is quasi-differentiable at x provided that there exists a convex, weak-star closed set $\partial f(x)$ such that, for all $d \in X$, the directional derivative $f'(x; d)$ of f at x in the direction d exists and*

$$f'(x; d) = \max_{x^* \in \partial f(x)} \langle x^*, d \rangle.$$

Evidently a convex function is quasi-differentiable at points where it is continuous and the quasi-differential coincides with the convex subdifferential; any function is quasi-differentiable where it is Gateaux differentiable, and the quasi-differential consists of a single element – the Gateaux derivative– at such a point. A systematic study of the properties of the quasi-differential and its applications in optimization problems was carried out in Pshenichnyĭ's monograph [54]. We will need the following special case of Theorem 4.2 of [54].

Theorem 6.2 *Let X be a Banach space and let C be a closed convex subset of X. Suppose that f is a locally Lipschitz quasi-differentiable function that attains a minimum at $x \in C$ over C. Then*

$$0 \in \partial f(x) + N(C, x),$$

where $N(C, x)$ is the convex normal cone of C at x.

Now we can state and prove our refined multidirectional mean value inequality for quasi-differentiable functions.

Theorem 6.3 *Let X be a Banach space with $x \in X$, let Y be a nonempty, closed and convex subset of X, and let $f : X \to \bar{R}$ be a locally Lipschitz quasi-differentiable function. Suppose that f is bounded below on $[x, Y]$ and*

$$\inf_{y \in Y} f(y) - f(x) > r.$$

Then, for any $\varepsilon > 0$, there exist $z \in [x, Y]$ and $z^* \in \partial f(z)$ such that

$$r < \langle z^*, y - x \rangle + \varepsilon \|y - x\| \text{ for all } y \in Y.$$

Further, we can choose z to satisfy

$$f(z) < \inf_{[x,Y]} f + |r| + \varepsilon.$$

Proof. 1. A special case. We begin by considering the special case when

$$\inf_{y \in Y} f(y) > f(x) \text{ and } r = -\varepsilon < 0.$$

Let $\bar{f} = f + \delta_{[x,Y]}$. Then $\bar{f}$ is bounded below on X. Without loss of generality we may assume that

$$\varepsilon < \inf_{y \in Y} f(y) - f(x).$$

Applying the Ekeland variational principle we conclude that there exists z such that

$$\bar{f}(z) < \inf \bar{f} + \varepsilon \tag{35}$$

such that

$$\bar{f}(w) \geq \bar{f}(z) - \varepsilon \|w - z\|. \tag{36}$$

That is to say

$$w \to f(w) + \delta_{[x,Y]}(w) + \varepsilon \|w - z\|$$

attains a minimum at z. By (35) $\bar{f}(z) < +\infty$. Therefore $z \in [x, Y]$. Applying Theorem 6.2 we have

$$0 \in \partial(f(\cdot) + \varepsilon \| \cdot - z\|)(z) + N([x, Y], z).$$

Noting that $\partial(f(\cdot) + \varepsilon \| \cdot - z\|)(z) \subset \partial f(z) + \varepsilon B_{X^*}$, there exists $z^* \in \partial f(z)$ such that

$$0 < \langle z^*, w - z \rangle + \varepsilon \|w - z\|, \quad \forall w \in [x, Y] \backslash \{z\}. \tag{37}$$

Moreover, inequality (35) implies that $\bar{f}(z) \leq f(x) + \varepsilon < \inf_Y f$, so $z \notin Y$. Thus we can write $z = x + \bar{t}(\bar{y} - x)$ where $\bar{t} \in [0, 1)$. For any $y \in Y$ set $w = y + \bar{t}(\bar{y} - y) \neq z$ in (37) yields

$$0 < \langle z^*, y - x \rangle + \varepsilon \|y - x\|, \quad \forall y \in Y. \tag{38}$$

2. The general case. We now turn to the general case. Consider $X \times R$ with the norm $\|(x,r)\| = \|x\| + |r|$. Take an $\varepsilon' \in (0, \varepsilon/2)$ small enough so that

$$\inf_{y \in Y} f(y) - f(x) > r + \varepsilon'$$

and define $F(z,t) := f(z) - (r+\varepsilon')t$. Obviously F is locally Lipschitz on $X \times R$ and is bounded below on $[(x,0), Y \times \{1\}]$. Moreover, one can directly check that F is quasi-differentiable and $\partial F(x,t) = \partial f(x) \times \{r + \varepsilon'\}$. Furthermore,

$$\inf_{Y \times \{1\}} F = \inf_{Y} f - (r+\varepsilon') > f(x) = F(x,0).$$

Applying the special case proved above with f, x and Y replaced by F, $(x,0)$ and $Y \times \{1\}$ we conclude that there exist $(z,s) \in [(x,0), Y \times \{1\}]$ and $z^* \in \partial f(z)$ satisfying

$$f(z) - (r+\varepsilon')s < \inf_{(w,t) \in [(x,0), Y \times \{1\}]} (f(w) - (r+\varepsilon')t) + \varepsilon'$$

i.e.,

$$\begin{aligned} f(z) &< \inf_{(w,t) \in [(x,0), Y \times \{1\}]} (f(w) - (r+\varepsilon')(t-s)) + \varepsilon' \\ &\le \inf_{[x,Y]} f + |r| + \varepsilon \end{aligned}$$

such that, for all $y \in Y$,

$$\begin{aligned} 0 &< \langle z^*, y - x \rangle - (r+\varepsilon') + \varepsilon'(\|y-x\| + 1) \\ &= \langle z^*, y - x \rangle - r + \varepsilon'\|y-x\| \le \langle z^*, y-x \rangle - r + \varepsilon\|y-x\|. \end{aligned}$$

This completes the proof. ■

Since continuous convex functions and Gateaux differentiable functions are quasi-differentiable, the following two corollaries follow directly from Theorem 6.3.

Corollary 6.4 *Let X be a Banach space with $x \in X$, let Y be a nonempty, closed and convex subset of X, and let $f : X \to R$ be a locally Lipschitz Gateaux differentiable function. Suppose that f is bounded below on $[x,Y]$ and*

$$\inf_{y \in Y} f(y) - f(x) > r.$$

Then, for any $\varepsilon > 0$, there exist $z \in [x, Y]$ such that

$$r < \langle \nabla f(z), y - x \rangle + \varepsilon \|y - x\| \text{ for all } y \in Y.$$

Further, we can choose z to satisfy

$$f(z) < \inf_{[x,Y]} f + |r| + \varepsilon.$$

Corollary 6.5 *Let X be a Banach space with $x \in X$, let Y be a nonempty, closed and convex subset of X, and let $f : X \to R$ be a convex continuous function. Suppose that f is bounded below on $[x, Y]$ and*

$$\inf_{y \in Y} f(y) - f(x) > r.$$

Then, for any $\varepsilon > 0$, there exist $z \in [x, Y]$ and $z^* \in \partial f(z)$ such that

$$r < \langle z^*, y - x \rangle + \varepsilon \|y - x\| \text{ for all } y \in Y.$$

Further, we can choose z to satisfy

$$f(z) < \inf_{[x,Y]} f + |r| + \varepsilon.$$

6.2 The Robinson-Ursescu open mapping theorem

We give an alternative proof of this theorem using the refined multidirectional mean value inequality, Corollary 6.5. A multifunction $F : X \to 2^Y$ is called a closed convex multifunction if the graph of F is a closed convex set. Recall that a set $S \subset X$ is absorbing provided that $X = \bigcup_{\lambda > 0} \lambda S$ and a point s is in the core of S (denoted by $s \in \text{core } S$) provided that $S - s$ is absorbing. The Robinson-Ursescu open mapping theorem says that when F is a convex multifunction, conditions in Theorem 4.1 can be replaced by $\bar{y} \in \text{core } F(X)$.

Theorem 6.6 (Robinson-Ursescu) *Let $F : X \to 2^Y$ be a closed convex multifunction. Suppose that $\bar{y} \in \text{core} F(X)$. Then F is open at $\bar{y}$, that is to say, for any $\bar{x} \in F^{-1}(\bar{y})$ there exists $\sigma > 0$ such that for $r > 0$ small enough,*

$$\text{int } B_{\sigma r}(\bar{y}) \subset \text{int } F(B_r(\bar{x})).$$

Proof. Let $p : X \times Y \to Y$ be a linear operator defined by $p(x, y) = y$. Since

$$p(\text{Graph } F - (\bar{x}, \bar{y})) = F(X) - \bar{y}$$

is absorbing and Graph F is convex a standard catagory argument implies that there exists $\sigma > 0$ such that

$$2\sigma B_Y \subset \text{cl } p((\text{Graph } F - (\bar{x}, \bar{y})) \cap B_{X\times Y}). \tag{39}$$

We show that $B_{\sigma r}(p(\bar{x}, \bar{y})) \subset p(B_r(\bar{x}, \bar{y}) \cap \text{Graph } F)$. Let $z \in B_{\sigma r}(p(\bar{x}, \bar{y}))$ and set $h(x, y) := \|p(x, y) - z\|$. Applying Theorem 6.5 to function h, set $B_r(\bar{x}, \bar{y}) \cap \text{Graph } F$ and point $(\bar{x}, \bar{y})$ yields that there exist $u \in B_r(\bar{x}, \bar{y}) \cap \text{Graph } F$ and $u^* \in \partial h(u)$ such that

$$\inf_Y h - h(\bar{x}, \bar{y}) - \sigma r/2 \le \langle u^*, (x, y) - (\bar{x}, \bar{y})\rangle, \ \forall x \in B_r(\bar{x}, \bar{y}) \cap \text{Graph } F. \tag{40}$$

If $h(u) = 0$ then $p(u) = z$ and we are done. Otherwise $u^* = p^* y^*$ with $y^* \in \partial\|\cdot\|(p(u) - z)$ being an unit vector. Then we can rewrite (40) as

$$\begin{aligned} 0 &\le \inf_Y h \le h(\bar{x}, \bar{y}) + \sigma r/2 + \langle y^*, p((x, y) - (\bar{x}, \bar{y}))\rangle \\ &\le \sigma r + \sigma r/2 + \langle y^*, p((x, y) - (\bar{x}, \bar{y}))\rangle, \ \forall (x, y) \in B_r(\bar{x}, \bar{y}) \cap \text{Graph } F. \end{aligned}$$

Observing that $2\sigma r B_Y \subset \text{cl } p((\text{Graph } F - (\bar{x}, \bar{y})) \cap r B_{X\times Y})$ the infinimum of the right hand side of the above inequality is $-\sigma r/2$, a contradiction. ■

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