Chapter 4
Differentiation Rules

We consider in this chapter symbolic and numerical computations related to the differentiation rules. We also explore implicit functions and implicit differentiation a little more than the text does. This is loosely coordinated with Chapter 4 of the text *Calculus with Early Vectors*, by Phillip Zenor, Edward Slaminka, and Donald Thaxton, Prentice Hall, 1999.

Warning: *The fact that the TI-89 "knows" the differentiation rules does not relieve you of the responsibility to learn them for yourself.* Once you become fairly proficient at differentiation by hand, you can write down the derivative much faster than you can type in the function and have the calculator do it. Most instructors will want to test your knowledge of the rules of differentiation in the absence of the tool. Use the tool only as an aid while you are learning the rules.

1. Exploring Differentiation Rules with the Calculator

We explore various ways to see that the calculator knows the following rules:

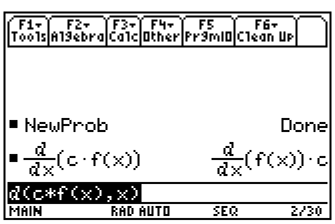
Theorem *Suppose that c is a real number, that f and g are real-valued functions so that f' and g' are defined at x, and that n is a positive integer. Then*

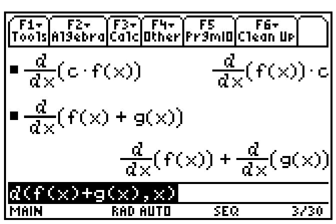
(*a*) $\left(c\,f\right)'(x) = c\,f'(x)$. *(Multiple rule)*

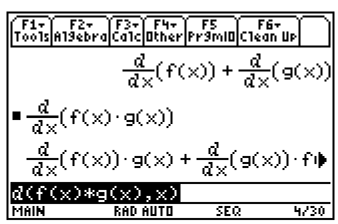
(*b*) $\left(f+g\right)'(x) = f'(x) + g'(x)$. *(Sum rule)*

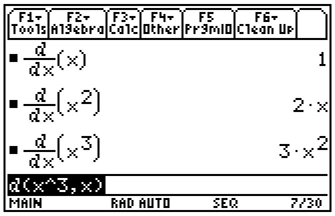
(*c*) $\left(f\,g\right)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$. *(Product rule)*

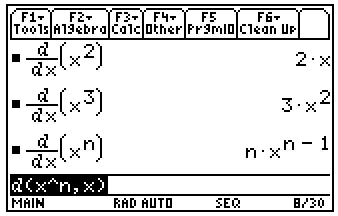
(*d*) $\dfrac{d\left(x^n\right)}{dx} = n\,x^{n-1}$

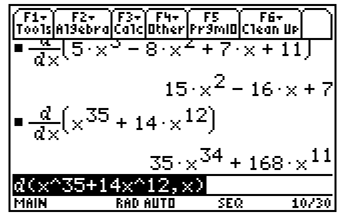












When we ask to the calculator to differentiate something undefined, it tends to indicate what it will do as far as it can. Effectively the "rules" above allow us to differentiate any polynomial. With a little practice, you can write down the derivative of a polynomial much faster than you can type in the original polynomial. You can use the calculator to check while you are first learning, but as soon as possible, you will want to loose your dependency upon the tool to do simple derivatives.

More interesting are the generalizations of the product rule for vector-valued functions.

Theorem *If h is a real-valued function and* $\vec{f}$ *and* $\vec{g}$ *are vector-valued functions defined on a common subset of* $\mathbb{R}$ *and differentiable at t, then*

(a) $D\left(\vec{f}\cdot\vec{g}\right)(t) = \vec{f}\,'(t)\cdot\vec{g}(t) + \vec{f}(t)\cdot\vec{g}\,'(t)$

(b) $D\left(\vec{f}\times\vec{g}\right)(t) = \vec{f}\,'(t)\times\vec{g}(t) + \vec{f}(t)\times\vec{g}\,'(t)$

(c) $D\left(h\,\vec{f}\right)(t) = h'(t)\,\vec{f}(t) + h(t)\,\vec{f}\,'(t)$

```
■ NewProb                                                    Done
■ Define f(t) = [5·t  t^3  sin(t)]                           Done
■ Define g(t) = [cos(t)  2·t  t^2]                           Done
■ d/dt(dotP(f(t), g(t)))            (t^2 + 5)·cos(t) - 3·t·sin(t) + 8·t^3
■ dotP(f(t), d/dt(g(t))) + dotP(d/dt(f(t)), g(t))
                                    (t^2 + 5)·cos(t) - 3·t·sin(t) + 8·t^3
■ d/dt(crossP(f(t), g(t)))
  [-2·t·cos(t) - 2·sin(t) + 5·t^4  2·(cos(t))^2 - 15·t^2 - 1  -3·t^2·cos(t) + t^3·sin(t) + 20·t]
■ crossP(d/dt(f(t)), g(t)) + crossP(f(t), d/dt(g(t)))
  [-2·t·cos(t) - 2·sin(t) + 5·t^4  2·(cos(t))^2 - 15·t^2 - 1  -3·t^2·cos(t) + t^3·sin(t) + 20·t]
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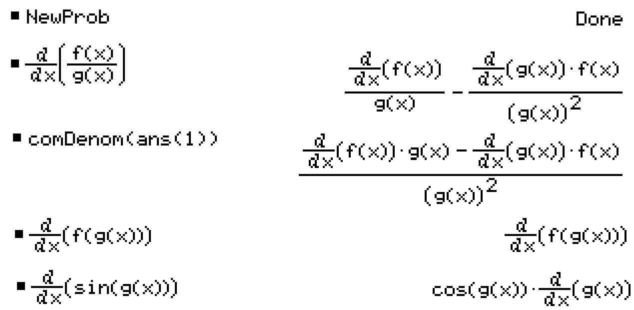
Certainly the calculator knows and can implement the quotient rule and the chain rule. It can actually "show you" the abstract quotient rule (using undefined functions) but it needs to know the "outer function" to proceed with the chain rule.

Theorem *If f and g are a real-valued functions defined on appropriate subsets of* $\mathbb{R}$ *and differentiable at t, then*

(a) $D\left(\dfrac{f}{g}\right)(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$ *(Quotient Rule)*

(b) $D\left(f\circ g\right)(x) = D\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$ *(Chain Rule)*

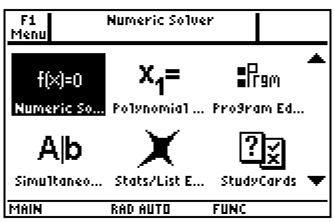


2. Implicit Functions and Plotting

Up until now, you have tended to only work with functions for which you have a simple explicit formula (such as can be easily typed into the Y= Editor). We now start to work with functions which may not have such a simply-found formula. Consider the set of all pairs (x, y) in the plane satisfying the equation

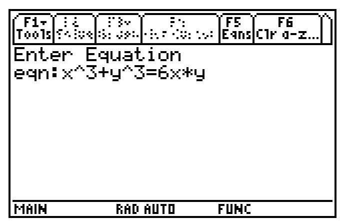
$$x^3 + y^3 = 6xy.$$

You can easily verify that the pair (3, 3) is in this set. We wish to claim that if you specify an x-value near 3, then there is a corresponding y-value (also near 3) that still solves the equation. We can use the numerical solver to see how this works for a few values.

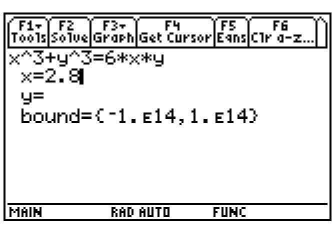


APPS Numeric Solver
(APPS Desktop ON)

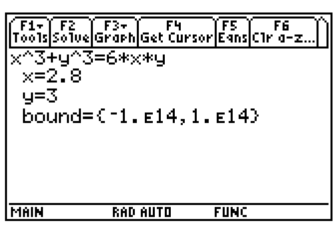
APPS Numeric Solver
(APPS Desktop OFF)



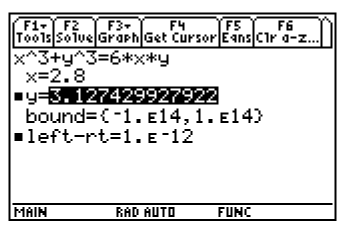
Enter equation



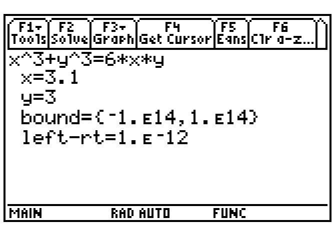
Enter desired x-value



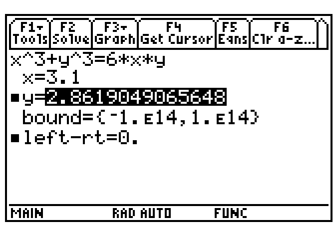
Enter “seed” y-value



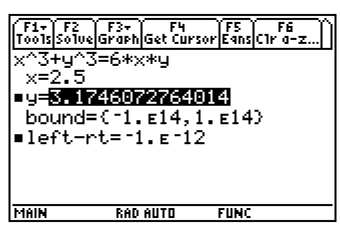
Press F2 Solve



Enter x-value and “seed”



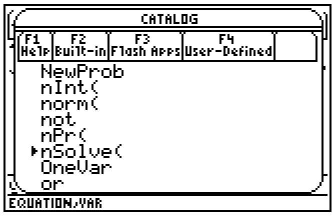
Press F3 Solve



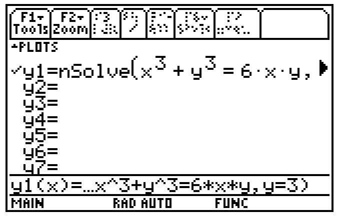
Once more

We say that the equation $x^3 + y^3 = 6xy$ implicitly defined a function near the solution pair (3, 3).

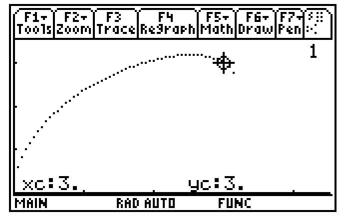
What we want to do now is to get a plot of this function on the calculator. The trick we use is to make use of a command line version of this numeric solver, which can be found in the catalog.



Command in CATALOG



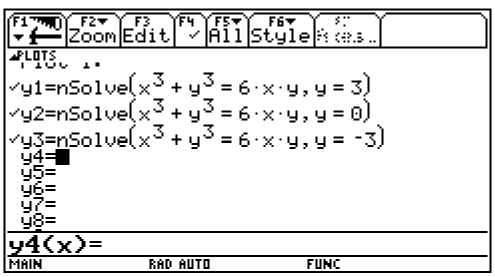
"Seed" is last argument as "y=3"



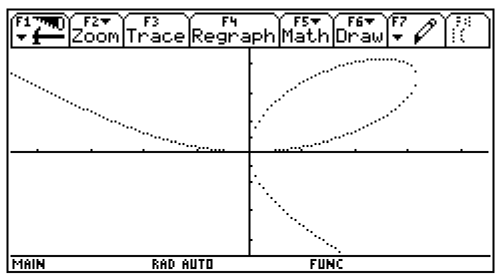
$0 \le x \le 4.5$, $0 \le y \le 3.5$,
Press TRACE, type $x = 3$

It is strongly suggested that you do not make your initial viewing window to large. Also since it takes a very long time to do this plotting, select *xres* = 2 or 3 to reduce the number of points plotted (and the time). Finally select the DOT style because this function might not be defined for the whole *x*-range to give (the one above is not) and the LINE style might "drop down" to another implicit function defined by the same equation.

To get a larger plot of the solution set for this equation, we expand the viewing window a little and put in some additional numeric solver commands with different seeds.



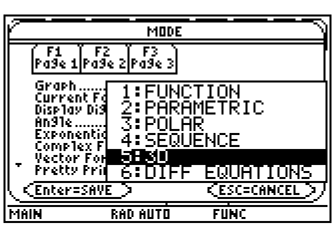
Voyage 200 screen, Y= Editor



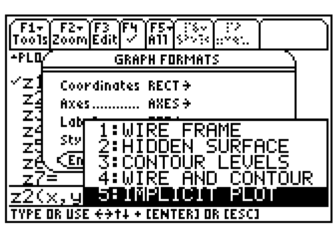
$-4.5 \le x \le 4.5$, $-3.5 \le y \le 3.5$, *xres* = 2 (plots slowly!)

From the plot, we can guess that for $x = 3$ there will actually be three different solutions for possible *y*-values which solve the equation. We can go back to the interactive numeric solver and try these alternate seeds to find that $y = 3$ or 1.8541019662496 or –4.8541019662499.

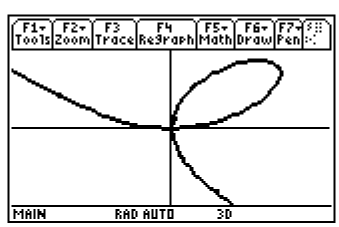
The TI-89 has implicit plotting as a style in 3D graphing mode, but you cannot easily add the tangent line or trace along the curve (as we will want to do in the next section), so the above seems better for a MATH 122 class.



Select 3D graphing



z1=x^3+y^3–6x*y F1 Format Style



$-4.5 \le x \le 4.5$, $-3.5 \le y \le 3.5$

3. Implicit Differentiation

Once you get a rough understanding of implicitly-defined functions, then you can ask for how we can find the derivative for such a function. It turns out that this usually ends up to be implicitly-defined as well. For illustration purposes, we go through the steps for what is called implicit differentiation for $x^3 + y^3 = 6xy$.

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{3\left(2y - x^2\right)}{3\left(y^2 - 2x\right)} = \frac{2y - x^2}{y^2 - 2x}$$

In particular, since we know that (3, 3) solves the original equation (giving us the "top" function in our plot), then we can compute the derivative there.

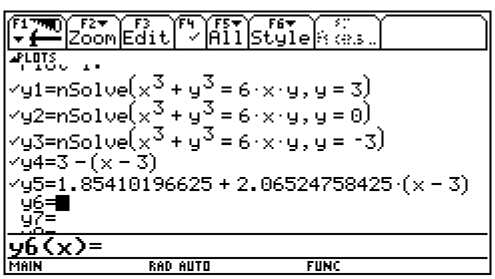
$$\left.\frac{dy}{dx}\right|_{x=3,\,y=3} = \frac{2(3) - (3^2)}{(3^2) - 2(3)} = -1$$

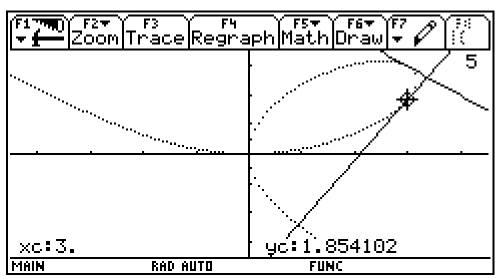
In a similar fashion, we can find the derivatives at the other solutions corresponding to $x = 3$.

$$\left.\frac{dy}{dx}\right|_{x=3,\,y=1.8541019662496} = \frac{2(1.8541019662496\ ) - (3^2)}{((1.8541019662496)^2) - 2(3)} = 2.06524758425$$

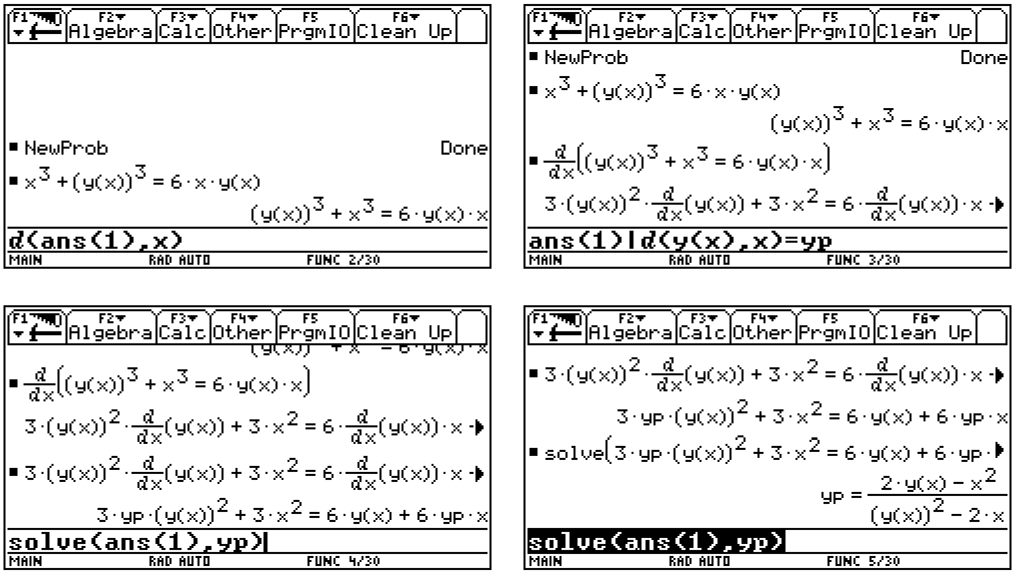
$$\left.\frac{dy}{dx}\right|_{x=3,\,y=-4.8541019662499} = \frac{2(-4.8541019662499\ ) - (3^2)}{((-4.8541019662499)^2) - 2(3\ \ )} = -1.06524758425$$

We now add the tangent lines (for the "top" function and the "middle" function) to the plot we have of this curve.

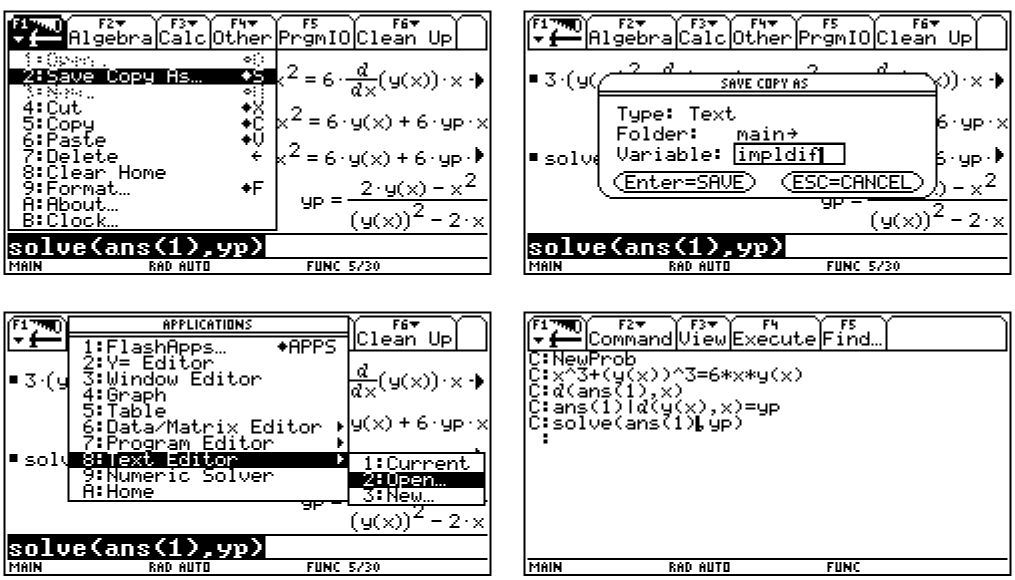




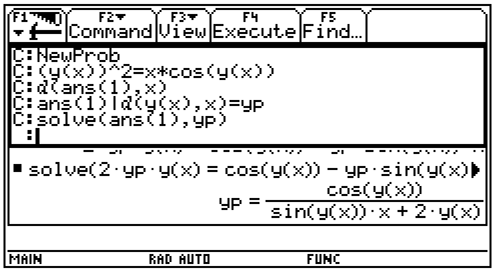
Although it might be more trouble than it is worth, it is possible to get the calculator to do symbolic implicit differentiation. At least you can use it to confirm some of your hand work as you are first learning to do implicit differentiation. To accomplish this, we need to convince the calculator that *y* is some unknown function of *x*. We do this by replacing the single letter *y* by the notation *y*(*x*). The result of differentiating such an equation has the derivative of this unknown function *y*(*x*) in it. Unfortunately we can "solve" for such an unknown derivative in the result. However we can substitute a new variable name (we use *yp* below) for this derivative. Then we solve for *yp*.

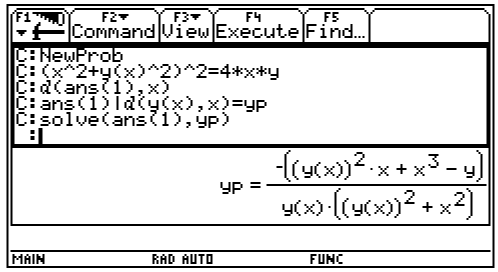


A tricky process like the above steps for implicit differentiation might be something that you want to save in a text file. Notice that everywhere that we used the previous answer, the actual answer has been inserted in the command as it appears in the history. You will have to edit the text file to put the command ANS(1) back in. Then you can simply edit the original equation and run the text file as a script to perform another implicit differentiation.



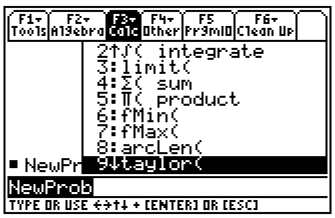
To work a new problem, (1) edit the original equation (second line of the text file), (2) move the cursor up to the first line, (3) press F3 View 1:Script view to split the screen to see both the text file at the top and the home screen at the bottom, and (4) press F4 Execute four times to complete the implicit differentiation.

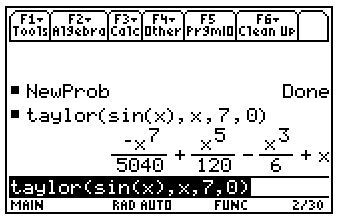


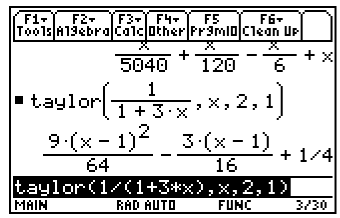


4. Taylor Polynomials

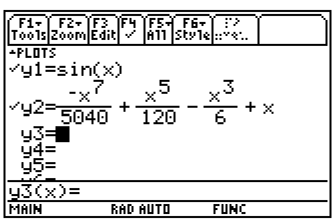
There is a nice command to generate a Taylor polynomial of any order for a function centered at any point.

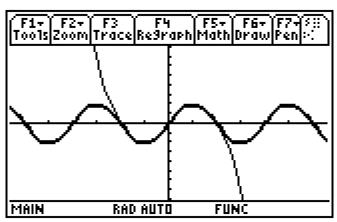






To improve the speed of plotting, generate the polynomials in the home screen , copy and paste the results in the Y= Editor for graphing. If you put the taylor command in the Y= Editor, the polynomial is re-generated again for every point plotted.





F2 Zoom 7:ZoomTrig