Chapter 5
Exploring Functions and Curves

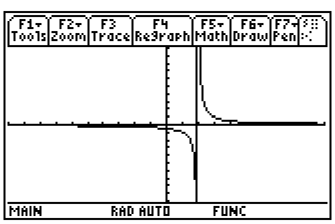
§1. Zooming to See Asymptotes
§2. Points of Inflection
§3. Maxima and Minima
§4. Tangential and Normal Components of Acceleration
§5. Circular Motion

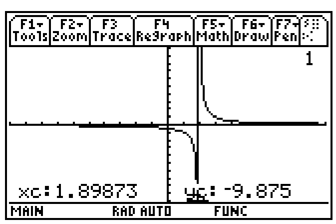
For the activities in this chapter, the various graphing capabilities of the calculator will be used. This is loosely coordinated with Chapter 5 of the text *Calculus with Early Vectors*, by Phillip Zenor, Edward Slaminka, and Donald Thaxton, Prentice Hall, 1999.

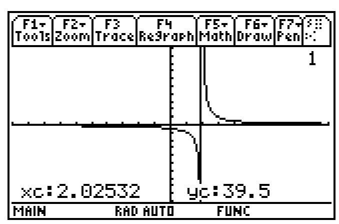
1. Zooming to See Asymptotes

Asymptotes can roughly be described as observing how the graph of a function behaves as something goes to infinity. Since the viewing window is always finite in every direction, calculator and computer graphs have a difficult time accurately representing either vertical or horizontal asymptotes. In particular, the two pieces of a vertical asymptote (where the function is not continuous) may be connected by a near vertical line segment which makes the plot appear continuous. Remember that in the default style (line), the calculator plot evaluates the function at a finite number of points, plots these points in the graph, and then connects the points (in order) with line segments. If the two plotted points lie on different sides of a vertical asymptote, then we would like to remove the line segment.

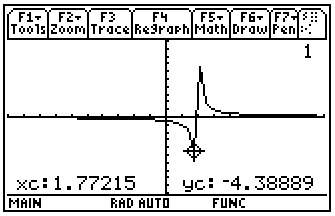
Consider $f(x) = \dfrac{1}{x-2}$ for viewing window $-10 \le x \le 10, \quad -10 \le y \le 10$ (ZoomStd).



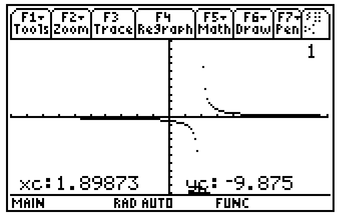




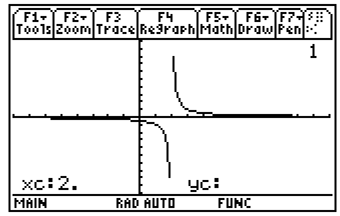
If we press F3 Trace and move with the cursor keys to the plotted point with the largest *x* less than 2, we find the plotted point (1.89873, –9.875) near the bottom of the viewing window. Moving with the right cursor key once to the smallest *x* greater than 2, we find the plotted point (2.02532, 39.5) well above the viewing window. The near vertical line in the plot attempts to connect these two points. All of the above plots have xres = 1. Making xres larger than 1 can give a plot looking even worse with the connecting line segment looking less vertical (see below).



xres = 3 Style Line



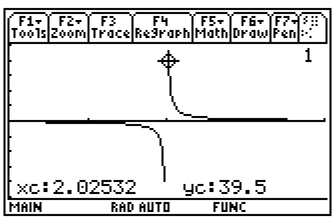
xres = 1 Style Dot



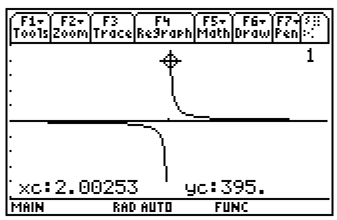
$-8 \le x \le 12, -10 \le y \le 10$

Notice above how the Style Dot removes all of the line segments between plotted points on the graph. I usually find this less satisfying (but it certainly removes the “false” near vertical line segment we do not want as well as all others the we would really still like to have). If the plot actually attempts to plot at $x = 2$, as it will do in the window $-8 \le x \le 12, -10 \le y \le 10$, then it find a place where the function is undefined (see above). No connecting line segments will be drawn across such a place where the function is know to be undefined.

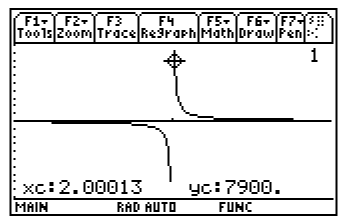
The most convincing way to visualize a vertical asymptote is to zoom in near the *x*-value where this happens at the same zooming out to see larger and larger *y*-values. In a similar fashion, we need to zoom out along the *x*-axis and zoom in along the *y*-axis to better understand a horizontal asymptote.



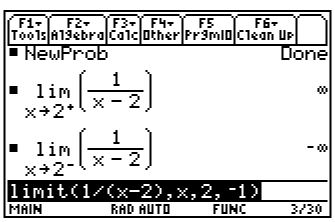
$0 \le x \le 4, -50 \le y \le 50$

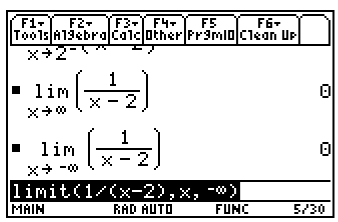


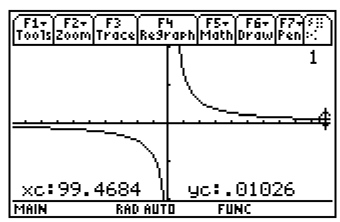
$1.8 \le x \le 2.2, -500 \le y \le 500$



$1.99 \le x \le 2.01, -10,000 \le y \le 10,000$

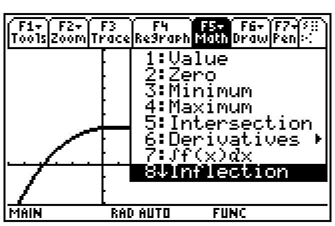


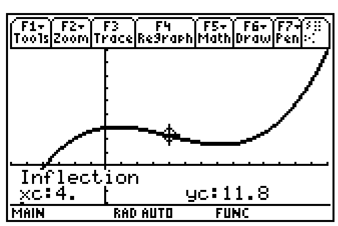




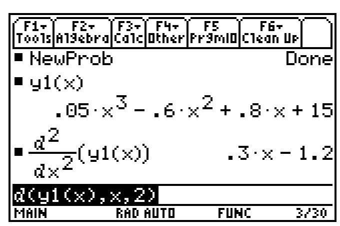
$-98 \le x \le 102, -0.2 \le y \le 0.2$

2. Points of Inflection

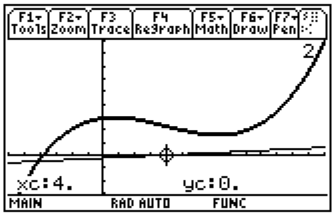




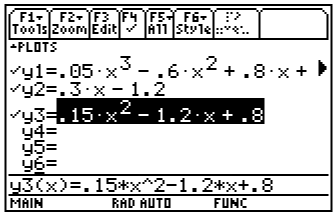
$-6 \le x \le 14, -15 \le y \le 45$

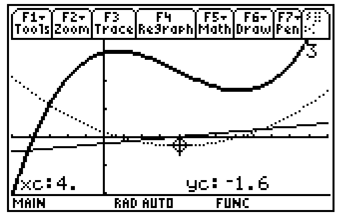


We can find points of inflection numerically and graphically in the graph screen, and we can work with the formulas symbolically in the home screen to solve exactly for where the second derivative is zero.



$-6 \le x \le 14, -15 \le y \le 45$



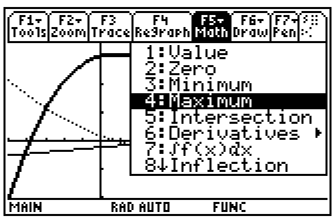


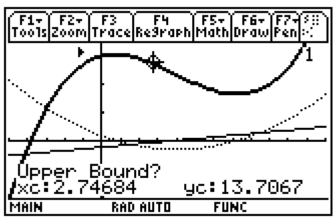
$-5 \le x \le 12, -10 \le y \le 17$

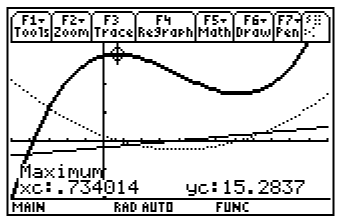
Further we can plot the function together with its derivative and/or its second derivative to help visualize how the sign of the second derivative near the *x*-value giving a point of inflection on the graph is related to concavity.

3. Maxima and Minima

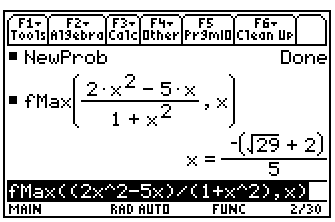
The graphical and numerical work with maxima and minima can take place in the graph screen.

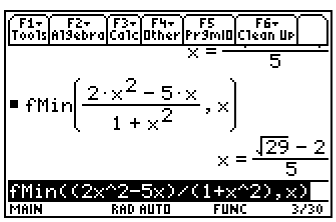






Symbolic work can take place in the home screen.





4. Tangential and Normal Components of Acceleration

For vector-valued functions such as $\mathbf{r}(t) = [\, f(t), g(t) \,]$, we have worked with the velocity vector $\mathbf{v}(t) = [\, f'(t), g'(t) \,]$ and the acceleration vector $\mathbf{a}(t) = [\, f''(t), g''(t) \,]$ before. We now consider how the calculator can be very useful for symbolically computing the tangential and normal components of the acceleration. We also look at how to visualize these vectors as line segments in the parametric plot of the image of the vector-valued function.

New Prob	Done
Define r(t)=[5cos(t),2sin(t)]	Done

```
d(r(t),t)                                                   [-5 sin(t)   3 cos(t)]
ans(1)→v(t)                                                                  Done
d(r(t),t,2)                                                 [-5 cos(t)   -3 sin(t)]
ans(1)→a(t)                                                                  Done
Define at(t)=(dotP(a(t),v(t))/dotP(v(t),v(t)))*v(t)                          Done
Define an(t)=a(t)-at(t)                                                      Done
```

■ at(t) $\left[\dfrac{-80\cdot(\sin(t))^2\cdot\cos(t)}{16\cdot(\sin(t))^2+9} \quad \dfrac{48\cdot\sin(t)\cdot(\cos(t))^2}{16\cdot(\sin(t))^2+9}\right]$

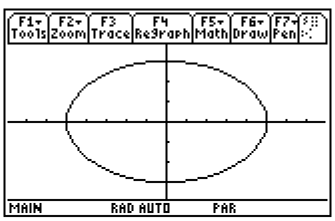
■ an(t) $\left[\dfrac{-45\cdot\cos(t)}{16\cdot(\sin(t))^2+9} \quad \dfrac{-75\cdot\sin(t)}{16\cdot(\sin(t))^2+9}\right]$

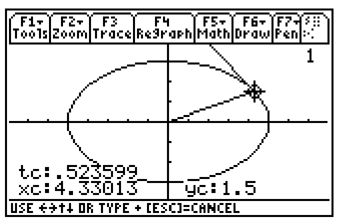
When we evaluate all of these functions at $t=\dfrac{\pi}{6}$, we get

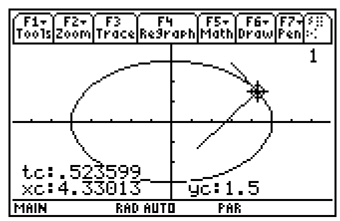
$$\mathbf{r}\left(\frac{\pi}{6}\right)=\left[\frac{5\sqrt{3}}{2},\frac{3}{2}\right],\ \ \mathbf{v}\left(\frac{\pi}{6}\right)=\left[-\frac{5}{2},\frac{3\sqrt{3}}{2}\right],\ \ \mathbf{a}\left(\frac{\pi}{6}\right)=\left[-\frac{5\sqrt{3}}{2},-\frac{3}{2}\right]$$

$$\mathbf{a}_T\left(\frac{\pi}{6}\right)=\left[-\frac{10\sqrt{3}}{13},\frac{18}{13}\right],\ \text{ and }\ \mathbf{a}_N\left(\frac{\pi}{6}\right)=\left[-\frac{45\sqrt{3}}{26},-\frac{72}{26}\right].$$

Below we have plotted the image of this vector-valued function as parametric equations. Then we have issued Line commands from the home screen to see the vectors above at this point.







ZoomDec

Line $\dfrac{5\sqrt{3}}{2},\dfrac{3}{2},\dfrac{5\sqrt{3}}{2}-\dfrac{5}{2},\dfrac{3}{2}+\dfrac{3\sqrt{3}}{2}$

shows velocity vector

Line $\dfrac{5\sqrt{3}}{2},\dfrac{3}{2},0,0$

shows acceleration vector

Line $\dfrac{5\sqrt{3}}{2},\dfrac{3}{2},\dfrac{5\sqrt{3}}{2}-\dfrac{10\sqrt{3}}{13},\dfrac{3}{2}+\dfrac{18}{13}$

shows tangential component $\mathbf{a}_T$

Line $\dfrac{5\sqrt{3}}{2},\dfrac{3}{2},\dfrac{5\sqrt{3}}{2}-\dfrac{45\sqrt{3}}{26},\dfrac{3}{2}-\dfrac{72}{26}$

shows normal component $\mathbf{a}_N$

5. Circular Motion

In this section, we continue the computation of the previous section to find the *radius of curvature*, the *curvature*, and the *center of the osculating circle* for a given vector-valued function.

```
Define rho(t)=dotP(v(t),v(t))/norm(an(t))                                    Done
Define kappa(t)=1/rho(t)                                                     Done
Define c(t)=r(t)+(dotP(v(t),v(t))/dotP(an(t),an(t)))*an(t)                   Done
```

Continuing with the example from the previous section $\mathbf{r}(t) = [5\cos(t), 3\sin(t)]$, we get

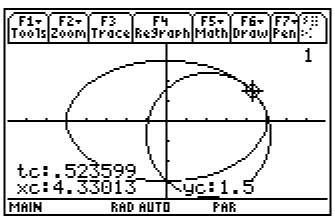
$$\rho(t) = \frac{\left(16(\sin(t))^2 + 9\right)^{3/2}}{15}, \quad \kappa(t) = \frac{15}{\left(16(\sin(t))^2 + 9\right)^{3/2}}, \text{ and}$$

$$c(t) = \left[\left(16/5 - \frac{16(\sin(t))^2}{5}\right)\cos(t), \frac{-16(\sin(t))^3}{3}\right].$$

Evaluated at $t = \frac{\pi}{6}$, we get

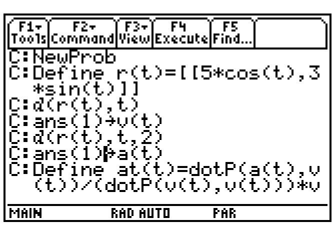
$$\rho\left(\frac{\pi}{6}\right) = \frac{13\sqrt{13}}{15}, \quad \kappa\left(\frac{\pi}{6}\right) = \frac{15\sqrt{13}}{169}, \text{ and } c\left(\frac{\pi}{6}\right) = \left[\frac{6\sqrt{3}}{5}, -\frac{2}{3}\right].$$

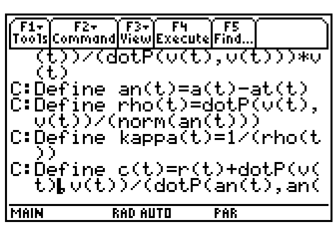
Finally we wish to plot the osculating circle in the graph of the image of the vector-valued function. We could work out the parametric equations for the circle, and put these in the Y= Editor. However if we have a "square" viewing window (as we should for this activity), then we can just draw the circle using a Circle command from the home screen using the center and the radius above.

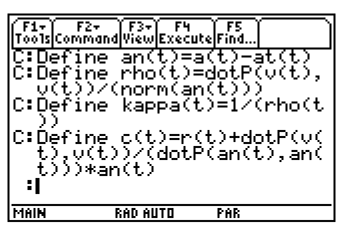


Circle $\frac{6\sqrt{3}}{5}, -\frac{2}{3}, \frac{13\sqrt{13}}{15}$

The whole process (from the definition of $\mathbf{r}(t)$ to the center of the osculating circle) is complicated enough that you would be well advised to save your "history" area after you have done this once as a text file. Then when you wish to do it again for a different vector-valued function, you can edit the original definition of $\mathbf{r}(t)$ in the text file and then execute the command again without typing so much. Where we have used the ans(1) command above, the text file from the history will have the actual answer in our example. You will also need to edit these lines in the text editor to put ans(1) back in. The final text file should look something like the following figures.







Thus to do this all again for a new problem, you need to edit the definition of r(t) in the second command line. Remember that you can F4 Execute the commands in the text file (those lines with a C: at the beginning) one at a time. Since for many problems, you will simply want to do all of this a fast as possible, it might be preferable to F2 Command 5: Execute to EOF after making sure the cursor was in the first command line in the text file.