ECE 6560
Multirate Signal Processing
Chapter 1

Dr. Bradley J. Bazuin
Western Michigan University
College of Engineering and Applied Sciences
Department of Electrical and Computer Engineering
1903 W. Michigan Ave.
Kalamazoo MI, 49008-5329
Why Multirate Filters?

• In digital signal processing, we want to operate as efficiently as possible at the lowest data rate that is reasonable.
  – Sample at a high rate, process and reduce the rate as quickly as possible.
  – Multirate processing provides implementation methods with the mathematical theory to validate them.

• Tools for real-time software radios and signal processing.
  – Identical performance, software upgrades/reprogrammability, multiple applications

• As processors have become faster, analog challenges can be traded off with DSP
Examples Where MRSP is Used

- CD digital-to-analog conversion

- Analog-to-digital conversion anti-aliasing
Compact Disc DAC

Figure 1.1 Signal Conditioning Tasks Required to Convert a Digital Signal to its Analog Representation

- Data Samples
- Digital-to-Analog Conversion
- Sample and Hold
- Analog Low Pass Filter
Time Signals

• Signal Representations

Figure 1.2 Three Time Signals: Input and Samples of Input, DAC Output, and Filtered Output

Fourier Transform Domain

- Equivalent Power Spectra

![Diagram of Fourier Transform Domain](image-url)
Concerns

• DAC-ZOH based filtering and distortion
  – Sinc function - spectral roll-off should be compensated
  – Spectral replica elements remain at sinc sidelobe maxima

• Analog LPF filter – fix distortion, attenuate stopband
  – Shape-factor = stopband BW / passband BW
  – Stopband attenuation requirement
  – Must specify filter passband (inverse distortion), transition band, and stop band!

• Is there a better way?
Supplemental Material

• It is assumed that you all understand Fourier Transforms
  – Fourier Transform pairs (continuous time/frequency)

• It is assumed that you all understand the material from ECE 4550/5550, Digital Signal Processing.
  – Fourier Transform pairs (discrete time/frequency)
  – Sampling theory (both sampling and reconstruction)
Class Questions … Example

• Write down the continuous time mathematical expressions for the original system block diagram elements.
  – time domain: impulse train … rect convolution in time … analog filter (convolution)
  – frequency domain: sample rate based replication … sinc function multiplication …. filter spectrum multiplication

• Based on multiplication and convolution operations, describe why the power spectrum appears as it does.

• Material that helps follows …
Filtering

- Convolution
  \[ y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) \cdot d\tau \]
  \[ y(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n - m) \]

- Preferred equation, showing causality if \( h(\tau) = 0 \) for \( \tau < 0 \)
  \[ y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) \cdot d\tau \]
  \[ y(t) = \int_{0}^{\infty} h_{causal}(\tau) \cdot x(t - \tau) \cdot d\tau \]
  \[ y(n) = \sum_{m=-\infty}^{\infty} h(m) \cdot x(n - m) \]
  \[ y(n) = \sum_{m=0}^{\infty} h_{causal}(m) \cdot x(n - m) \]

Note: an FIR filter is causal or can be made causal
Sampling and ZOH

• Using delta functions

\[ x_s(t) = d(t) = \sum_n \delta(t - n \cdot T_s) \cdot x(t) \]

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• Zero Order Hold

\[ ZOH(t) = \text{rect} \left( \frac{t - \frac{T_s}{2}}{T_s} \right) \]

\[ x_{ZOH}(t) = d_{ZOF}(t) = \left[ \text{rect} \left( \frac{t - \frac{T_s}{2}}{T_s} \right) \right] * \left[ \sum_n \delta(t - n \cdot T_s) \cdot x(n \cdot T_s) \right] \]
Continuing

\[
x_{ZOH}(t) = d_{ZOF}(t) = \int_{-\infty}^{\infty} \text{rect} \left( \frac{\tau - T_s}{T_s} / 2 \right) \cdot \left[ \sum_n \delta(t - \tau - n \cdot T_s) \cdot x(n \cdot T_s) \right] \cdot d\tau
\]

\[
x_{ZOH}(t) = d_{ZOF}(t) = \int_{0}^{T_s} \left[ \sum_n \delta(t - \tau - n \cdot T_s) \cdot x(n \cdot T_s) \right] \cdot d\tau
\]

\[
x_{ZOH}(t) = d_{ZOH}(t) = \sum_n \cdot x(n \cdot T_s) \cdot \int_{0}^{T_s} \delta(t - \tau - n \cdot T_s) \cdot d\tau
\]

\[
x_{ZOH}(t) = d_{ZOH}(t) = x(n \cdot T_s), \quad \text{for } 0 \leq t - n \cdot T_s < T_s
\]

- Continuous time description of sampled process
What is the Spectrum?

- Multiplication in time domain by sampling function
  - Convolve signal spectrum with frequency sampling function

\[ X_s(f) = D(f) = \sum_n \delta(f - \frac{n}{T_s}) \ast X(f) \]

- Replicated spectrum based on the sample rate! (aliasing)
- Applying the ZOH “filter”

\[ X_{ZOH}(f) = D_{ZOH}(f) = X_s(f) \cdot \Re\left[ \text{rect}\left( \frac{t - 0.5 \cdot T_s}{T_s} \right) \right] \]

\[ X_{ZOH}(f) = D_{ZOH}(f) = X_s(f) \cdot T_s \cdot \text{sinc}(f \cdot T_s) \cdot \exp(j \cdot \pi \cdot f) \]

- Fundamental band frequency roll-off due to sinc. This causes the distortion observed in the desired passband!
Therefore the Steps Produced

- **Equivalent Power Spectra**

  Notice spectral droop at the frequency band-edge of the passband.

Analog compensation filter: eliminate droop and reduce residual spectral replicas.

*Figure 1.3 Three Spectra: Sampled Input, DAC Output, and Filtered Output*
To finish the processing …

- If the audio maximum frequency is 20 kHz and the sample rate is 44.1 kHz, what do we need for a low pass filter?
  - Spectral audio band -20kHz to +20kHz with replicas at 44.1kHz +/- 20kHz and -44.1kHz +/- 20kHz (others too, but not important for the LPF)
  - LPF passband to 20 kHz, stopband at (44.1-20=) 24.1kHz
  - Shape factor is 24.1/20 = 1.205 … not a reasonable value

- Shape factor rule-of-thumb
  - SF > 3 LC for RF, op-amp based for audio
  - 1.28 < SF < 3 SAW, crystal or other special filter
  - SF < 1.28 Analog very expensive, not practical
Alternate Compact Disc: 4:1 Oversampling

Figure 1.4 Modified Signal Conditioning Tasks Required to Convert Digital Signal to Analog Representation

- Interpolate
- Digital Filter
- Digital-to-Analog Conversion (higher rate)
- Sample and Hold
- Analog Low Pass Filter
Time Signals

- Signal Representations

![Diagram showing time signals and their transformations](Image)

**Figure 1.5** Four Time Signals: Input, Samples and 1-to-4 Interpolated Samples of Input, 1-to-4 Up Sampled DAC Output, and Filtered Output
Frequency Domain

- Equivalent Power Spectra

Digital compensation filter: eliminate droop and reduce residual spectral replicas.

DAC with ZOH at higher sampling rate

Analog compensation filter: same functions, much easier requirements

Figure 1.6 Four Spectra: Sampled Input, 1-to-4 Up Sampled Output, DAC Output, and Filtered Output
New Analog filter

• If the audio maximum frequency is 20 kHz and the sample rate is 176.4 kHz, what do we need for a low pass filter?
  – Spectral audio band -20kHz to +20kHz with replicas at 176.4kHz +/- 20kHz and -176.4kHz +/- 20kHz (other too, but not important for the LPF)
  – LPF passband to 20 kHz, stopband at (176.4-20=) 156.4kHz
  – Shape factor is 156.41/20 = 7.82

• Shape factor rule-of-thumb
  – SF > 3  LC for RF, op-amp based for audio
  – 1.28 < SF < 3  SAW, crystal or other special filter
  – SF < 1.28  Analog very expensive, not practical
Result

• Significantly easier analog filter specifications using additional digital signal processing
  – What is the “production cost” tradeoff?

• Up side
  – Analog filter spec is simplified and can be cheap
    (any filter that is close will do – cheap parts, no hand-touch labor!)
  – DAC/ZOH sinc function distortion reduced and you can include a digital pre-compensation in the digital filter

• Down side
  – Requires digital filtering and signal processing
  – DAC operates at 4x previous rate
Example 2: Anti-Aliasing Filter

Figure 1.7 Signal Conditioning Tasks Required to Convert an Analog Signal to its Digital Representation

- Analog Signal Input
- Anti-aliasing filter
- Automatic Gain Control
- Sample and Hold
- Analog-to-digital Converter
Frequency Domain

- Power Spectrum

**Figure 1.8** Spectra of Input Signal, Analog Filtered Analog Signal, and Sampled Input Signal
Alternate Anti-Aliasing Filter

Figure 1.9. Modified Signal Conditioning Tasks Required to Convert an Analog Signal to its Digital Representation

- Analog Signal Input
- Anti-aliasing filter
- Automatic Gain Control
- Sample and Hold (higher rate)
- Analog-to-digital Converter (higher rate)
- Digital Filter
- Decimated Output
Frequency Domain

- Power Spectrum

Figure 1.10 Four Spectra: Filtered Input, 4-Times Oversampled Sampled Input, Digitally Filtered Output, and 4-to-1 Down Sampled Output
Result

• Significantly easier analog filter specifications using additional digital signal processing
  – What is the “production cost” tradeoff?

• Up side
  – Analog filter spec is simplified and can be cheap
    (any filter that is close will do – cheap parts, no hand-touch labor!)
  – Parallel channels can have “identical” frequency and phase characteristics

• Down side
  – Requires digital filtering and signal processing
  – ADC operates at 4x previous rate
Telephone Audio Example

- Telephone audio is roughly 300 Hz to 3.2 kHz and is typically sampled at 8 ksps.
  - Replicated bands
    - 0 kHz +/- 3.2 kHz (-3.2 to + 3.2)
    - 8 kHz +/- 3.2 kHz (8-3.2 to 8+3.2 or 4.8 to 11.2)
    - -8 kHz +/- 3.2 kHz (-8-3.2 to -8+3.2 or -11.2 to -4.8)
    - Etc.
  - LPF passband: 3.2 kHz
  - LPF stopband: 4.8 kHz (8-3.2 kHz … not 8/2 kHz)
  - SF at 8 ksps = 4.8/3.2 = 1.5 needs a special filter!

Applying similar technique:
- SF at 4 x 8 = 32 ksps = (32-3.2)/3.2 = 9 easy filter!
Where are Multirate Filters Used?

- Any and all ADC application with anti-aliasing filters
- Any and all DAC applications with output filters

- Audio Signal Processing
  - Telephones (all), recorded music (CD, MP3, etc), movie sound, etc.

- All Modern Wireless Systems
  - Transmitters and receivers
  - Advanced modulation schemes
Signal Processing
Knowledge Elements

• Signal Sampling
  – Digital data rates and how they can be changed without destroying the signal content

• Filters
  – Analog definitions, digital equivalent generation, or all-digital
  – Windows and filters

• Transmitting communication symbols

• Polyphase Filters
  – Filter Decimation and Interpolation Filter

• Filter Bank analysis (filter decimation) and synthesis (interpolation filter)

• Special Filters: Half-Band, CIC, All-Pass

Matlab Overview

• Step through Chap1.m
  – i2pi function
  – dB and dBv
  – FFT function and matrix multiplication
  – Built in filter functions: butterworth and elliptical

• Useful for estimating filter orders: ButterPlot
  – Butterworth filters is where the focus on the “3 dB” point comes from!!
  – Analog and RF designers do not focus on 3 dB ….
Class Question …

• Do you need a Fourier Transform Review?
  – Derivations
  – Symmetry
  – Implementation
  – “Special aspects” that will be regularly applied
    • IFFT of “perfect frequency domain filters”
    • “Windowing” of time samples → convolution in frequency domain
    • Spectral symmetry
    • Spectral inversion (Fs/2 shifting) and other shifting
    • The Odd-Frequency Fourier Transform (OFFT)

  – My bet is yes …