Partial Velocities and the Slider Crank Mechanism

System Configuration

The figure shows a simple slider crank mechanism with no offset. Given the physical dimensions of the links \((R, L)\), the configuration of the system at any instant of time can be given by one or all of the generalized coordinates \((q_i) = (\theta, \phi, x)\). These coordinates are not independent, so we can write a set of constraint equations, such as

\[
\begin{align*}
RS_\theta - LS_\phi &= 0 \\
RC_\theta + LC_\phi - x &= 0
\end{align*}
\] (1)

Partial Angular Velocities of the Links

Using the angles shown in the diagram, the angular velocities of the crank and connecting bar can be written as \(\omega_{AB} = \dot{\theta}k\) and \(\omega_{BC} = -\dot{\phi}k\). From these results, we can define two obvious partial angular velocities.

\[
\begin{align*}
\frac{\partial \omega_{AB}}{\partial \dot{\theta}} &= k \\
\frac{\partial \omega_{BC}}{\partial \dot{\phi}} &= -k
\end{align*}
\] (2)

But if we differentiate the first of Eq. (1), we can see that

\[
R\dot{\theta}C_\theta = L\dot{\phi}C_\phi
\] (3)

So, we can define another set of partial angular velocities as

\[
\begin{align*}
\frac{\partial \omega_{AB}}{\partial \dot{\phi}} &= \frac{\partial}{\partial \dot{\phi}} \left( \dot{\theta}k \right) = \frac{\partial}{\partial \dot{\phi}} \left[ \left( \frac{LC_\phi}{RC_\theta} \right) \dot{k} \right] = \left( \frac{LC_\phi}{RC_\theta} \right) k \\
\frac{\partial \omega_{BC}}{\partial \dot{\theta}} &= \frac{\partial}{\partial \dot{\theta}} \left( -\dot{\phi}k \right) = -\frac{\partial}{\partial \dot{\theta}} \left[ \left( \frac{RC_\theta}{LC_\phi} \right) \dot{k} \right] = -\left( \frac{RC_\theta}{LC_\phi} \right) k
\end{align*}
\] (4, 5)
Partial Velocities of the Slider

The velocity of the slider may be written most simply as \( v_c = \dot{x} \). From this result, we can define the partial velocity

\[
\frac{\partial v_c}{\partial \dot{x}} = \dot{i}
\]  

(6)

However, we can also write the velocity of the slider in terms of the crank or connecting bar angular rates as follows:

\[ v_c = v_B + v_{C/B} \]

where

\[ v_B = v_{B/A} = \omega_{AB} \times \ell_{B/A} = \dot{k} \times R(C_\theta \dot{i} + S_\theta \dot{j}) = R\dot{\theta}(-S_\theta \dot{i} + C_\theta \dot{j}) \]

\[ v_{C/B} = \omega_{BC} \times \ell_{C/B} = -\dot{\phi} \times L(C_\phi \dot{i} - S_\phi \dot{j}) = -L\dot{\phi}(S_\phi \dot{i} + C_\phi \dot{j}) \]

So,

\[ v_c = (-R\dot{\theta}S_\theta - L\dot{\phi}S_\theta) \dot{i} + (R\dot{\theta}C_\theta - L\dot{\phi}C_\theta) \dot{j} = (-R\dot{\theta}S_\theta - L\dot{\phi}S_\theta) \dot{i} \]  

(7)

Note that the \( \dot{j} \) component is zero as a result of Eq. (3). Using Eqs. (3) and (7), we can now define the following partial velocities:

\[ v_c = -\dot{\theta} \left[ RS_\theta + Ls_\phi \left( \frac{RC_\theta}{LC_\phi} \right) \right] \dot{i} \]  

\[
\frac{\partial v_c}{\partial \dot{\theta}} = -R \left[ S_\theta + C_\theta \frac{S_\phi}{C_\phi} \right] \dot{i}
\]  

(8)

\[ v_c = -\dot{\phi} \left[ RS_\theta \left( \frac{LC_\phi}{KC_\theta} \right) + Ls_\phi \right] \dot{i} \]  

\[
\frac{\partial v_c}{\partial \dot{\phi}} = -L \left[ S_\phi + C_\phi \frac{S_\theta}{C_\theta} \right] \dot{i}
\]  

(9)