The closed loop control system shown below has a digital compensator represented by the discrete transfer function $G_c(z)$ and a continuous plant represented by the continuous transfer function $G_p(s)$. The digital control signal ($u(kt)$) is shown to be converted into a continuous signal $u(t)$ using a zero-order hold (ZOH).

**Block Diagram of a Continuous/Discrete Closed-Loop System**

- **One common method of finding $G_c(z)$ involves a two step process.**
  - First, design its continuous counterpart $G_c(s)$ using continuous compensator design techniques (root locus and/or Bode diagrams)
  - Then transform the resulting transfer function into a discrete form

Some authors refer to this as *emulation*, because the discrete compensator emulates its continuous counterpart.

- **Tustin’s approximation** is one common method used to make this transformation. In this method, the substitution $s = \frac{2}{T} \left(\frac{z-1}{z+1}\right)$ is made into the continuous transfer function to find $G_c(z)$.

- For example, the continuous phase lead compensator
  \[ G_c(s) = \frac{3(s+5)}{s+15} \]

becomes

\[
G_c(z) = \frac{3 \left( \frac{2}{T} \left( \frac{z-1}{z+1} \right) + 5 \right)}{\frac{2}{T} (z-1)+15(z+1)} = \frac{3 \left( \frac{2}{T} (z-1)+5(z+1) \right)}{\left(15 + \frac{6}{T}\right)z + \left(15 - \frac{6}{T}\right)}.
\]
For a sample time $T = 0.001$ (sec), we get
\[
G_c(z) = \frac{6015z - 5985}{2015z - 1985} = \frac{2.9851z - 2.97}{z - 0.9851}
\]

It can be shown that when Tustin’s approximation is applied to an integral compensator, the trapezoidal rule is used to approximate the integral.

MATLAB’s “c2d” command can also be used to convert from continuous to discrete transfer functions. The methods of conversion include the Tustin approximation and a zero-order hold on the input to the transfer function.

```
>> num = 3*[1,5];
>> den = [1,15];
>> sys = tf(num,den)
Transfer function:
3 s + 15
------
 s + 15
>> sysD = c2d(sys,0.001,'tustin')
Transfer function:
2.985 z - 2.97
--------------
z - 0.9851
Sampling time: 0.001
```

Given the discrete transfer function, the compensator can be written as a difference equation as follows.

\[
G_c(z) = \frac{U(z)}{E(z)} = \frac{6015z - 5985}{2015z - 1985} = \frac{2.985z - 2.97}{z - 0.9851}
\]

or

\[(z - 0.9851)U(z) = (2.985z - 2.97)E(z)\]

Multiply by $z^{-1}$ and solving for $U(z)$

\[(1 - 0.9851z^{-1})U(z) = (2.985 - 2.97z^{-1})E(z)\]

or

\[U(z) = 0.9851z^{-1}U(z) + 2.985E(z) - 2.97z^{-1}E(z)\]
This last result is equivalent to the difference equation

\[ u(k) = 0.9851 u(k-1) + 2.985 e(k) - 2.97 e(k-1) \]

Another common method is the MPZ (matched pole-zero) Method.

As with any numerical method, it provides an approximation of the original continuous transfer function. The accuracy of the approximation is usually application dependent.

The MPZ method is based on mapping the poles and zeros of the continuous transfer function using the relationship \( z = e^{sT} \) and preserving the low frequency gain.

For example, if we have a phase lead or lag type compensator of the form

\[ G_c(s) = K \left( \frac{s+a}{s+b} \right) \]

then, the discrete equivalent for a given sample time \( T \) is

\[ G_c(z) = K' \left( \frac{z-e^{-aT}}{z-e^{-bT}} \right) \]

Here, \( K' \) is found by applying the final value theorems to each transfer function and equating the results.

\[ \lim_{s \to 0} (s \cdot \frac{1}{s} \cdot G_c(s)) = K \left( \frac{a}{b} \right) = \lim_{z \to 1} \left( \frac{1-z^{-1}}{1-z^{-1}} \cdot G_c(z) \right) = K' \left( \frac{1-e^{-aT}}{1-e^{-bT}} \right) \]

or

\[ K' = K \left( \frac{a}{b} \right) \left( \frac{1-e^{-bT}}{1-e^{-aT}} \right) \]

More details on this method may be found in Franklin, Powell, and Emami-Naeini, \textit{Feedback Control of Dynamic Systems}, Prentice-Hall, 2002.