Fig. 1 shows a spring-mass-damper (SMD) system with a **force actuator** for position control. The spring has stiffness $k$, the damper has coefficient $c$, the block has mass $m$, and the position of the mass is measured by the variable $x$.

The transfer function of the SMD with the actuating force $F_a$ as input and the position $x$ as output is

$$\frac{X(s)}{F_a} = \frac{1}{ms^2 + cs + k}$$

Assuming ideal actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, $X_d$ represents the **desired position**, and $G_c(s)$ represents the **transfer function** of the controller.

For the following analyses, it is assumed the SMD parameters are: $m=1$ slug, $c=8.8$ (lb-s/ft), and $k=40$ (lb/ft). This represents an **under-damped, second-order plant** with

$$\omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)}$$

$$\zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7$$
Proportional Control

- For proportional control, \( G_c(s) = K \), and the loop and closed-loop transfer functions are

\[
GH(s) = \frac{K}{s^2 + 8.8s + 40} \quad \frac{X}{X_d}(s) = \frac{K}{s^2 + 8.8s + (40 + K)}
\] (2)

- Using \( GH(s) \), the RL diagram for the closed-loop system for \( K \geq 0 \) is shown in Fig. 2. Note that as the value of \( K \) is increased, the closed-loop poles move straight up/down, indicating that the natural frequency is increased and the damping ratio is decreased as \( K \) is increased.

- This is a type-zero system and hence will have a finite steady-state error for a step input. Using the final-value theorem and the closed-loop transfer function, \( x_{ss} \) the final value of \( x(t) \) to a unit step command is

\[
x_{ss} = \lim_{s \to 0} \left( s \cdot \frac{1}{s} \cdot \frac{K}{s^2 + 8.8s + (40 + K)} \right) = \frac{K}{40 + K} < 1
\] (3)

- Eq. (3) indicates that large values of \( K \) lead to small steady-state error; however, they also lead to a faster, less damped responses.

- This conclusion is verified in Fig. 3 which shows the closed-loop step responses for gains \( K \) of 100, 500, and 2000. Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. To remove the steady-state error and have better response, integral and/or derivative terms must be included.
Proportional-Integral (PI) Control

- For proportional-integral (PI) control: \[ G_c(s) = K_p + \frac{K_i}{s} = \frac{K_p(s + a)}{s} \] (4)

- Here, \( K_p \) and \( K_i \) represent the proportional and integral gains, and \( a = K_i/K_p \) is the ratio of the integral and proportional gains. In this case, the loop and closed-loop transfer functions are

\[
GH(s) = \frac{K_p(s + a)}{s(s^2 + 8.8s + 40)} \quad \frac{X}{X_d}(s) = \frac{K_p(s + a)}{s(s^2 + 8.8s + 40) + K_p(s + a)}
\] (5)

- Using integral control makes the system type-one, so the steady-state error due to a step input is zero. This can be verified using the final value theorem to show that \( x_{ss} = 1 \) when the input is a unit step function.

- Fig. 4 shows the RL diagram for the closed-loop system with \( a = 3 \). It also shows the location of the closed-loop poles for a proportional gain \( K_p \approx 50 \). Fig. 5 shows the closed-loop step response for \( a = 3 \) and \( K_p = 25, 50, \) and 75.

- Integral control has removed the steady-state error and improved the transient response, but it has also increased the system settling time.

Figure 4. Root Locus Diagram for PI Control \((a = 3)\)

Figure 5. Step Response for PI Control \((a = 3)\)
for Various Proportional Gains
Proportional-Derivative (PD) Control

- For **proportional-derivative (PD) control**: 
  \[ G_c(s) = K_p + K_D s = K_D(s + a) \]  
  \( (6) \)

- \( K_p \) and \( K_D \) represent the **proportional** and **derivative gains**, and \( a = K_F / K_D \) is the ratio of the proportional and derivative gains. The loop and closed-loop transfer functions are

  \[ \frac{GH(s)}{X_d(s)} = \frac{K_D(s + a)}{s^2 + 8.8s + 40} \]

  \[ X_d(s) = \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \]  
  \( (7) \)

- Without integral control, this is a **type-zero** system, and hence will have a **finite steady-state error** to a unit step input. Using the final-value theorem and the closed-loop transfer function, \( x_{ss} \) the final value of \( x(t) \) to a unit step command is

  \[ x_{ss} = \lim_{s \to 0} \left( s \cdot \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \right) = \frac{K_Da}{40 + K_Da} = \frac{K_P}{40 + K_P} < 1 \]  
  \( (8) \)

As with proportional control, the **larger the proportional gain**, the **smaller the steady-state error**.

- **Fig. 6** shows the RL diagram for the closed-loop system with \( a = 10 \). It also shows the location of the closed-loop poles for \( K_D \approx 25.6 \). **Fig. 7** shows the **closed-loop step response** for \( a = 10 \) and derivative gains of \( K_D = 10, 27, 50, \) and 75.

- The PD controller has **decreased the system settling time** considerably; however, to control the steady-state error, the derivative gain \( K_D \) must be high. This **decreases the response times** of the system and can make it **susceptible to noise**.

![Figure 6. Root Locus Diagram for PD Control (a = 10)](image)

![Figure 7. Step Response for PD Control (a = 10) for Various Derivative Gains](image)
Proportional-Integral-Derivative Control

- For proportional-integral-derivative (PID) control:
  \[
  G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s}
  \] (9)

- \( K_p, K_I, \) and \( K_D \) represent the proportional, integral, and derivative gains, \( a = \frac{K_p}{K_D} \) is the ratio of the proportional and derivative gains, and \( b = \frac{K_I}{K_D} \) is the ratio of the integral and derivative gains. In this case, the loop and closed-loop transfer functions are

\[
GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)}
\]

\[
\frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)}
\] (10)

- Again, with integral control, the system is type-one and has zero steady-state error for a step input.
- **Fig. 8** shows the RL diagram of the closed-loop system for \( a = 15 \) and \( b = 50 \). The location of the closed-loop poles for \( K_D \approx 15.8 \) is also shown.
- **Fig. 9** shows the step response of the closed-loop system for \( a = 15, b = 50, \) and various derivative gains.
- The PID controller has removed steady-state error and decreased system settling times while maintaining a reasonable transient response.

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Figure 8. Root Locus Diagram for PID Control
\((a = 15, b = 50)\)

Figure 9. Step Response for PID Control
\((a = 15)\) 
\((b = 50)\) for Various Derivative Gains