ME 555 Intermediate Dynamics
Angular Momentum and Kinetic Energy of a Simple Crank Shaft

The figure to the right shows a simple crank shaft consisting of seven segments, each considered to be a slender bar. Each segment of length \( \ell \) has mass \( m \). There are six segments of length \( \ell \) and one segment of length \( 2\ell \) (segment 4). The mass center of the system is \( G \) and is located on the axis of rotation.

The system is undergoing fixed axis rotation, so \( H_G \), the angular momentum of the system about its mass center is calculated as follows:

\[
\begin{align*}
H_G \cdot \mathbf{i}' & = \begin{bmatrix} I_{x'x'}^G & -I_{x'y'}^G & -I_{x'z'}^G \\ -I_{y'x'}^G & I_{y'y'}^G & -I_{y'z'}^G \\ -I_{z'x'}^G & -I_{z'y'}^G & I_{z'z'}^G \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} -I_{x'z'}^G \omega \\ -I_{y'z'}^G \omega \\ I_{z'z'}^G \omega \end{bmatrix} \\
\end{align*}
\]

where

\[
I_{zz}^G = \sum_{i=1}^{7} \left( I_{zz}^G \right)_i = 0 + \frac{1}{3} m \ell^2 + m \ell^2 + \frac{1}{12} (2m)(2\ell)^2 + m \ell + \frac{1}{3} m \ell^2 + 0 \\
= \frac{10}{3} m \ell^2
\]

\[
I_{x'z'}^G = \sum_{i=1}^{7} \left( I_{x'z'}^G \right)_i = 0 + m(\ell)(-\ell) + m(\ell)(-\ell) + 0 + m(-\ell)(\ell) + m(-\ell) = \frac{10}{3} m \ell^2
\]

\[
I_{y'z'}^G = 0 \quad \text{(since the } X'Z \text{ plane is a plane of symmetry)}
\]

So,

\[
H_G = 2m \ell^2 \omega \mathbf{i}' + \left( \frac{10}{3} m \ell^2 \omega \right) \mathbf{k}
\]

The kinetic energy of the crank shaft is found from the velocity and angular momentum vectors to be

\[
K = \frac{1}{2} m (\mathbf{\dot{r}}_G)^2 + \frac{1}{2} \mathbf{\dot{\omega}}_B \cdot H_G = \frac{1}{2} \mathbf{\dot{\omega}}_B \cdot H_G = \frac{1}{2} (\omega \mathbf{k}) \cdot H_G = \frac{10}{6} m \ell^2 \omega^2
\]