Some remarks on vertex Folkman numbers for hypergraphs

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Abstract

Let $F(r, G)$ be the least order of $H$ such that the clique number of $H$ and $G$ are equal and any $r$-coloring of the vertices of $H$ yields a monochromatic and induced copy of $G$. The problem of bounding of $F(r, G)$ was studied by several authors and it is well understood. In this note, we extend those results to uniform hypergraphs.

1 Introduction

The general Ramsey-type problem in graph theory involves determining the existence, and consequently the least order, of a graph which guarantees that a certain property holds. One variation of this general problem is to impose certain restrictions on the size of the clique that the graph is allowed to contain. Folkman extended Ramsey theory in this direction. The following problem, originally raised by Erdős and Hajnal [9], asks to construct a graph $H$ that does not contain a copy of $K_{n+1}$ such that in every coloring of its edges with two colors, there is a $K_n$, all of whose edges have the same color. Folkman [10] proved the existence of such $H$. The general case, for an arbitrary number of colors, $r$, instead of just two, was settled affirmatively by Nešetřil and Rödl [17].

An alternate problem is one where we consider coloring the vertices instead of the edges of $H$. Folkman also proved that there is a $K_{n+1}$-free graph such that any $r$-coloring of its vertices yields a monochromatic copy of $K_n$. In [7], the first author and Rödl considered the more general problem of determining $F(r, G)$, the least order of $H$ such that $\omega(H) = \omega(G)$ and any $r$-coloring of the vertices of $H$ yields a monochromatic and induced copy of $G$. Note that in addition to the condition that $G$ and $H$ have the same clique number, they also required that the monochromatic copy of $G$ be induced. Some special cases of this function were considered previously by several researchers (see, e.g., [14, 15, 16]). For example, in [7], the first author and Rödl proved that

$$F(r, K_n) \leq cn^2(\log n)^4$$

for some constant $c = c(r)$. (Base of all logarithms in this paper is $e$.) Recently, together with Rödl [6], we obtained a more general result. Conditioning on the clique number of $G$

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of order $n$ we showed that

$$F(r, G) \leq \frac{cn^3}{\omega(G)}(\log n)^5 \tag{1}$$

for some $c = c(r)$.

It seems natural to attempt to extend those results to hypergraphs and quite surprisingly, the methods applicable to graphs extend to hypergraphs.

A hypergraph $G$ is a pair $(V, E)$, where $V$ is a set of vertices, and $E \subseteq 2^V$ is a set of hyperedges. The order of a hypergraph is the size of its vertex set. A hypergraph $G = (V, E)$ is $k$-uniform if every edge $e \in E$ has cardinality exactly $k$. The clique number of $G$, denoted by $\omega(G)$, is the order of the largest clique contained in $G$.

Here, we generalize the previous results on vertex Folkman numbers to hypergraphs. Let $r$ be a given number of colors and $G$ be a $k$-uniform hypergraph of order $n$. We define the (induced) Folkman number $F(r, G)$ of $G$ to be the minimum order of a $k$-uniform hypergraph $H$ with $\omega(H) = \omega(G)$ such that every $r$ coloring of the vertices of $H$ yields a monochromatic and induced copy of $G$.

Here we show that the Folkman numbers for hypergraphs are almost quadratic.

**Theorem 1** For all natural numbers $r \geq 1$ and $k \geq 3$ there are constants $c$ and $C$ such that

$$cn^2 \leq \max \{ F(r, G) \} \leq Cn^2(\log n)^2,$$

where the maximum is taken over all $k$-uniform hypergraphs $G$ of order $n$.

Note that this upper bound is always better than the one for graphs (cf. (1)). With some additional work (see the approach taken in [8]) we can reduce the factor $n^2(\log n)^2$ to $n^2 \log n$ in Theorem 1. However, in this note, we only present the simpler argument.

It is not difficult to see that for some families of hypergraphs the corresponding Folkman numbers are linear in order of the hypergraphs. For example, let us consider the $k$-partite $k$-uniform hypergraph $K^k_{n, \ldots, n}$ (every partition class has $n$ vertices and every edge intersects every set of the partition in exactly one vertex). By Theorem 1 there is a $k$-uniform hypergraph $H$ with $\omega(H) = k$ such that any $r$-coloring of its vertices yields a monochromatic copy of $K^k_{1, \ldots, 1}$ (just one hyperedge). Replace every vertex $v$ in $H$ by a set of vertices $U_v$ of size $r(n - 1) + 1$ and each edge $\{v_1, \ldots, v_k\}$ in $H$ by the complete $k$-partite $k$-uniform hypergraph with partition classes $U_{v_1} \cup \cdots \cup U_{v_k}$. Denote the new hypergraph by $H^*$. Clearly, $\omega(H^*) = k$ and $|V(H^*)| = O(n)$ for fixed $r$ and $k$. Now we claim that $H^*$ yields a monochromatic copy of $K^k_{n, \ldots, n}$ for any $r$-coloring of its vertices. Indeed, fix an $r$-coloring of $H^*$ with colors $c_1, \ldots, c_r$. This coloring induces the following $r$-coloring of the vertices of $H$. Assign color $c_i$ to $v \in V(H)$ if at least $n$ vertices in $U_v$ are colored by $c_i$ (we resolve conflicts arbitrarily). Since in $H$ there is a monochromatic copy of $K^k_{1, \ldots, 1}$, there is a monochromatic copy of $K^k_{n, \ldots, n}$ in $H^*$.

The above example may suggest that the hypergraphs with small clique numbers have linear Folkman numbers (or much smaller than quadratic). We show that in general this is far from being true.

**Theorem 2** For all natural numbers $r \geq 1$ and $k \geq 3$ there are constants $c$ and $d$ such that for every $n$ there is a $k$-uniform hypergraph $G$ of order $n$ and clique number $\omega(G) \leq d$ such...
that
\[ F(r, G) \geq c n^{\frac{\log \log \ldots \log n}{\log \ldots \log n}}. \]

(The constant \(d\) in the above theorem is linear in \(k\).)

One can also consider a slightly different problem and take the asymptotic in number \(r\) of colors. We complement the previous results as follows.

**Theorem 3** For every \(k\) and \(n\) there is a constant \(C\) such that for any \(k\)-uniform hypergraph \(G\) of order \(n\) and any number \(r\) of colors
\[ F(r, G) \leq C r^2. \]

Recently, the first author and Mubayi [5] using a different terminology showed that in the special case when \(G\) is the complete \(k\)-uniform hypergraph
\[ F(r, K_k^n) \leq Cr^2 (\log r)^{\frac{1}{k-2}}, \]
where \(C = C(k, n)\).

In view of the current bounds on \(F(r, G)\), we would like to pose the following two problems.

**Question 1** Is there a family of hypergraphs \(\{G_n\}\) for which \(F(r, G_n)\) is asymptotically larger than \(n^2\)?

**Question 2** Is there a hypergraph \(G\) of a fixed order \(n\) such that \(F(r, G) = \Omega(r^2)\)?

## 2 Proof of the main theorem

In this section we prove Theorem 1.

### 2.1 Upper bound

We adapt the approach taken for graphs (see, e.g., [2, 7]) to hypergraphs. First we construct a \(k\)-uniform hypergraph \(H\) (as a certain model of random graphs) of the appropriate order. Then we show that \(\omega(H) = \omega(G)\). Finally, we prove that a hypergraph \(H\) randomly chosen from the space of all such hypergraphs asymptotically almost surely yields a monochromatic and induced copy of \(G\) for any \(r\) coloring of its vertices. However, note that in order to prove the upper bound, we only need to show the existence of one of such \(H\).

#### 2.1.1 Construction

First recall that a projective plane \(PG(2, q)\) is an incidence structure of a set \(P\) of points and a set \(L\) of lines such that:

(P1) any two points lie in a unique line,
It is known that for every prime power $q$ such incidence structure $PG(2, q)$ exists with $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$ (see, e.g., [11]).

Fix natural numbers $r$ and $k$. We will show that for any $k$-uniform hypergraph $\mathcal{G}$ of order $n$ there exists a $k$-uniform hypergraph $\mathcal{H}$ of order $Cn^2\log^2 n$, $C = C(k, r)$, such that $\omega(\mathcal{H}) = \omega(\mathcal{G})$ and any subhypergraph of $\mathcal{H}$ induced by a set of cardinality $\frac{1}{r}|V(H)|$ contains an induced copy of $\mathcal{G}$.

By Bertrand’s postulate there is a prime number $q$ such that

$$2rn \log n \leq q + 1 \leq 4rn \log n.$$  \hfill (2)

Let $PG(2, q)$ be a projective plane with a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines. We construct a $\mathcal{H}$ (as a certain model of random graphs) with the vertex set $\mathcal{P}$. Clearly, $|V(\mathcal{H})| = q^2 + q + 1 = \Theta(n^2(\log n)^2)$.

For each line $\ell$ we choose one ordered partition $\ell_1, \ldots, \ell_n$, $x \leq |\ell_i| \leq x + 1$, $1 \leq i \leq n$, randomly and uniformly from the set of all such partitions. For simplicity, we assume that

$$|\ell_1| = |\ell_2| = \cdots = |\ell_n| = x = \frac{q + 1}{n}.$$ \hfill (3)

Let $V(\mathcal{G}) = \{v_1, \ldots, v_n\}$. For each $u_{i_1} \in \ell_{i_1}, u_{i_2} \in \ell_{i_2}, \ldots, u_{i_k} \in \ell_{i_k}$ we join $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ by a hyperedge if and only if $\{v_1, v_2, \ldots, v_k\} \in E(\mathcal{G})$. Note that $\mathcal{H}$ is well-defined because of condition (P1).

### 2.1.2 Clique number of $\mathcal{H}$

We now show that every clique in $\mathcal{H}$ is entirely contained in some line. Since each line induces a subhypergraph with clique number $\omega(\mathcal{G})$, it follows that $\omega(\mathcal{H}) = \omega(\mathcal{G})$.

Let $e = \{u_1, \ldots, u_k\}$ and $f = \{w_1, \ldots, w_k\}$ be two distinct hyperedges of a clique $\mathcal{K}$ in $\mathcal{H}$. Consider a tight path from $u_1$ to $w_k$, namely, the set of edges

$$\{u_1, u_2, \ldots, u_k, w_1\}, \{u_2, \ldots, u_k, w_1, w_2\}, \ldots, \{u_k, w_1, \ldots, w_{k-1}\}, \{w_1, \ldots, w_k\}.$$

It is an easy observation that all these hyperedges are in $\mathcal{K}$. Indeed, any two consecutive hyperedges share at least 2 vertices (recall that $k \geq 3$). Thus, by construction (and (P1)) any two consecutive hyperedges (and consequently all of them) belong to the same line. In particular, $e$ and $f$ lie in the same line.

**Remark 1** Note that the above argument does not work for graphs (when $k = 2$). Keeping the right clique number in $\mathcal{H}$ is somewhat more complicated for graphs (for more details see [6, 7]).

### 2.1.3 Arrowing $\mathcal{G}$

For simplicity we assume that $\frac{1}{r}|\mathcal{P}|$ is an integer. For $U \subseteq V(\mathcal{H})$ with $|U| = \frac{1}{r}|V(\mathcal{H})| = \frac{1}{r}|\mathcal{P}|$ let $A(U)$ be the event that $\mathcal{G}$ is not a subgraph of $\mathcal{H}[U]$. Clearly, $A(U)$ implies $A(\ell \cap U)$ for each $\ell \in \mathcal{L}$. Consequently,

$$A(U) \subseteq \bigcap_{\ell \in \mathcal{L}} A(\ell \cap U),$$

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and since all events $A(\ell \cap U)$ are independent (cf. (P1)),
\[ \Pr(A(U)) \leq \prod_{\ell \in L} \Pr(A(\ell \cap U)). \]  
(4)

For a fixed line $\ell \in L$ we bound from above the probability that $A(\ell \cap U)$ occurs. Note that if $A(\ell \cap U)$ occurs, then for some $i = i(\ell)$, $1 \leq i \leq n$, in partition $\ell = \bigcup_{j=1}^n \ell_j$, $\ell_i \cap U = \emptyset$. Let $|U \cap \ell| = u_\ell$. The probability that for a fixed $i$, $1 \leq i \leq n$, $U \cap \ell_i = \emptyset$ equals to the probability that for a fixed partition $\ell = \bigcup_{i=1}^n \ell_i$ randomly chosen subset $T$ of $\ell$ with $|T| = u_\ell$ satisfies $T \cap \ell_i = \emptyset$. Hence, the union bound implies that
\[ \Pr(A(\ell \cap U)) \leq n \left( \frac{q+1-x}{q+1} \right) \frac{u_\ell}{u_\ell} \leq n \exp \left( -\frac{xu_\ell}{q+1} \right), \]
where the last inequality holds since for any natural numbers $a, b, c$ satisfying $a - b \geq c$
\[ \frac{\binom{a-b}{c}}{\binom{a}{c}} = \frac{(a-b)-(c-1)}{a-(c-1)} \cdots \frac{a-b}{a} \leq \left( \frac{a-b}{a} \right)^c \leq \exp \left( -\frac{bc}{a} \right). \]

Consequently, (4) yields,
\[ \Pr(A(U)) \leq n|L| \exp \left( -\frac{x}{q+1} \sum_{\ell \in L} u_\ell \right). \]

Also, since every point in $U$ belongs to exactly $q + 1$ lines
\[ \sum_{\ell \in L} u_\ell = \sum_{\ell \in L} |U \cap \ell| = |U|(q + 1). \]

Hence,
\[ \Pr(A(U)) \leq n|L| \exp \left( -x|U| \right) = n|P| \exp \left( -x|U| \right). \]

This implies that
\[ \Pr \left( \bigcup_U A(U) \right) \leq \left( \frac{|P|}{1/\gamma} \right) n|P| \exp \left( -x|U| \right) \leq (er)^{1/\gamma} n|P| \exp \left( -x|U| \right), \]
where the union is taken over all subsets $U \subseteq V(H)$ with cardinality $1/\gamma|P|$. Finally, by (2) and (3) we get
\[ x|U| \geq 2r \log n \frac{|P|}{r} = 2|P| \log n, \]
and consequently,
\[ \Pr \left( \bigcup_U A(U) \right) \leq \exp \left( |P| (- \log n + o(\log n)) \right) = o(1). \]

Thus,
\[ \Pr \left( \bigcap_U \bar{A}(U) \right) = 1 - o(1) \]

which implies that almost every $H$ has a copy of $G$ in each subset of $1/\gamma|V(H)|$ vertices.
2.2 Lower bound

We define the union of two vertex-disjoint hypergraphs $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ to be the hypergraph $\mathcal{G} \cup \mathcal{G}' = (V \cup V', E \cup E')$. Let $\mathcal{K}^k_n$ denote the complete $k$-uniform hypergraph of order $n$ and $\overline{\mathcal{K}}^k_n$ be its complement (i.e. empty hypergraph).

Suppose $\mathcal{G} = \mathcal{K}^k_\omega \cup \mathcal{K}^k_{n-\omega}$ (so, $\omega(\mathcal{G}) = \omega$). Let $\mathcal{H}$ be a hypergraph such that any 2-coloring of its vertices yields a monochromatic copy of $\mathcal{G}$. Then $\mathcal{H}$ must contain many copies of $\mathcal{G}$ and so $\mathcal{K}^k_\omega$. Starting with $\mathcal{H}$, we remove copies of $\mathcal{K}^k_\omega$ successively until we can find no more copies of $\mathcal{K}^k_\omega$. Suppose we removed $\ell$ copies of $\mathcal{K}^k_\omega$. We color the vertices of the $\ell$ copies of $\mathcal{K}^k_\omega$ red and the remaining blue. Clearly, in this red-blue coloring, $\mathcal{H}$ does not contain a blue copy of $\mathcal{G}$, which implies that $\mathcal{H}$ must contain a red copy of $\mathcal{G}$. Since every red induced copy of $\mathcal{K}^k_{n-\omega}$ can have at most $k-1$ vertices from each copy of $\mathcal{K}^k_\omega$, the number of copies, $\ell$, of $\mathcal{K}^k_\omega$ in $\mathcal{H}$ must be at least $\frac{(n-\omega)\omega}{k-1}$, and consequently

$$|V(\mathcal{H})| \geq \ell \omega \geq \frac{(n-\omega)\omega}{k-1}.$$  

In particular, if $\omega(\mathcal{G}) = \Theta(n)$ and $n - \omega(\mathcal{G}) = \Theta(n)$ (e.g. $\mathcal{K}^k_{n/2} \cup \overline{\mathcal{K}}^k_{n/2}$), then

$$F(\mathcal{G}, r) \geq F(\mathcal{G}, 2) = \Omega(n^2)$$

for every fixed $r \geq 2$.

3 Another lower bound

In this section we prove Theorem 2.

Recall that the hypergraph Ramsey number $R_k(s, t)$ is the minimum $n$ such that every red-blue coloring of $\mathcal{K}^k_n$ contains a red copy of $\mathcal{K}^s_n$ or a blue copy of $\mathcal{K}^t_n$.

3.1 3-uniform hypergraphs

First we consider 3-uniform hypergraphs ($k = 3$). Due to a result of Conlon, Fox and Sudakov [4] it is known that

$$R_3(4, t) \geq 2^{ct \log t}. \quad (5)$$

Thus, there is a 3-uniform hypergraph of order $n$ which contains no clique of size 4 and no independent set of size bigger than $O\left(\frac{\log n}{\log \log n}\right)$. Let us denote such a graph by $\mathcal{R}^3_n$. Let $G^3_n = \mathcal{R}^3_{n/2} \cup \overline{\mathcal{R}}^3_{n/2}$. Clearly, $\omega(G^3_n) = 3$ and so the clique number is as small as possible.

We use a similar approach as in Section 2.2. Suppose $\mathcal{H}$ is a hypergraph such that any 2-coloring of its vertices yields a monochromatic copy of $G^3_n$. Then $\mathcal{H}$ must contain many copies of $G^3_n$. Starting with $\mathcal{H}$, we remove copies of $\mathcal{R}^3_{n/2}$ successively until we can find no more copies of $\mathcal{R}^3_{n/2}$. Suppose we removed $\ell$ such copies, colored the vertices of the $\ell$ copies of $\mathcal{R}^3_{n/2}$ red and the remaining blue. Then, in this red-blue coloring, $\mathcal{H}$ does not contain a blue copy of $G^3_n$. Hence, $\mathcal{H}$ must contain a red copy of $G^3_n$ and specifically, a red copy of
Since every red induced copy of $K_{n/2}^3$ can have at most $O(\log n \log \log n)$ vertices from each red copy of $R_{n/2}^3$, it follows that

$$\ell = \Omega \left( \frac{n \log \log n}{\log n} \right).$$

Thus,

$$|V(H)| \geq \ell|V(R_{n/2}^3)| = \Omega \left( \frac{n^2 \log \log n}{\log n} \right).$$

**Remark 2** This lower bound together with Theorem 1 shows that

$$F(r, G_n^3) = \Theta(n^{2+o(1)}).$$

It is worth mentioning that for graphs ($k = 2$) a similar approach gives a much worse result. Since $R_2(3, t) = \Theta(t^2 \log t)$ [1, 13], there is a $K_3$-free graph $R_n$ of order $n$ with independence number no bigger than $O(\sqrt{n \log n})$. Let $H$ be a graph for which any 2-coloring of its vertices induces a monochromatic copy of $R_{n/2} \cup K_{n/2}$. Then the above argument only shows that the order of $H$ is at least $\Omega(n^{3/2} \log \log n)$. On the other hand, (1) bounds from above the order of $H$ by $O(n^3 (log n)^5)$.

### 3.2 $k$-uniform hypergraphs for $k \geq 4$

Applying the refined version of the Conlon, Fox and Sudakov [3] stepping-up lemma, originally provided by Erdős and Hajnal (see, e.g., Section 4.7 in [12]), to (5) one can obtain that for $s = \lceil \frac{5}{2} k \rceil - 3$,

$$R_k(s, t) \geq 2^{ct \log t}$$

where the tower of 2 is of height $k - 2$. Thus, there is a $k$-uniform hypergraph of order $n$ which contains no clique of size $O(1)$ and no independent set of size bigger than

$$O\left( \frac{\log \log \ldots \log n}{\log \log \ldots \log \log n} \right).$$

Let us denote such graph by $R_n^k$. Let $G_n^k = R_n^k \cup K_n^k$. Clearly, $\omega(G_n^3) = O(1)$. Using an analogous approach as for $k = 3$ yields the statement.

**Remark 3** Note that for $k \geq 4$ it is not clear if

$$\max F(r, G) = \Omega \left( n^{2 \frac{\log \log \ldots \log n}{\log \log \ldots \log \log n}} \right),$$

where the maximum is taken over all $k$-uniform hypergraphs with clique number $s - 1 = k$. Because of the stepping-up lemma one has to assume that $s \geq \lceil \frac{5}{2} k \rceil - 3$. 

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4 Asymptotic in the number of colors

In this section we prove Theorem 3. The proof is basically a variation of the proof of Theorem 1.

Here we modify the previous construction. Fix natural numbers \( k \) and \( n \). We will show that for any \( k \)-uniform hypergraph \( G \) of order \( n \) there exists a \( k \)-uniform hypergraph \( H \) of order \( Cr^2 \), \( C = C(k, n) \), such that \( \omega(H) = \omega(G) \) and any subhypergraph of \( H \) induced by a set of cardinality \( \frac{1}{r}|V(H)| \) contains an induced copy of \( G \).

We choose a prime number \( q \) such that
\[
n^2r \leq q + 1 \leq 2n^2r.
\]

Let \( PG(2, q) \) be a projective plane with a set \( \mathcal{P} \) of points and a set \( \mathcal{L} \) of lines. We construct an \( H \) with the vertex set \( \mathcal{P} \). Clearly, \( |V(H)| = q^2 + q + 1 = \Theta(r^2) \).

As before we assume that \( n \) divides \( q + 1 \). For each line \( \ell \) we choose uniformly at random one ordered partition into \( n \) sets of the same size. Let
\[
|\ell_1| = |\ell_2| = \cdots = |\ell_n| = x = \frac{q + 1}{n}.
\]
Let \( V(G) = \{v_1, \ldots, v_n\} \). For each \( u_{i_1} \in \ell_{i_1}, u_{i_2} \in \ell_{i_2}, \ldots, u_{i_k} \in \ell_{i_k} \) we join \( \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\} \) by an edge if and only if \( \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G) \). Note that \( H \) is well-defined because of condition (P1).

It follows immediately from Section 2.1.2 that \( \omega(H) = \omega(G) \).

It remains to show that a hypergraph \( H \) randomly chosen from the space of all such hypergraphs asymptotically almost surely yields a monochromatic and induced copy of \( G \) for any \( r \) coloring of its vertices. We mimic the proof from Section 2.1.3.

For \( U \subseteq V(H) \) with \( |U| = \frac{1}{r}|V(H)| = \frac{1}{r}|\mathcal{P}| \) let \( A(U) \) be the event that \( G \) is not a subgraph of \( H[U] \). As in Section 2.1.3 one can show that
\[
\Pr\left(\bigcup_U A(U)\right) \leq (er)^{\frac{1}{2}|\mathcal{P}|n|\mathcal{P}|} \exp\left(-x|U|\right),
\]
where the union is taken over all subsets \( U \subseteq V(H) \) with cardinality \( \frac{1}{r}|\mathcal{P}| \). By (6) and (7) we get \( x|U| \geq n|\mathcal{P}| \). Thus,
\[
\Pr\left(\bigcup_U A(U)\right) \leq (er)^{\frac{1}{2}|\mathcal{P}|n|\mathcal{P}|} \exp\left(-n|\mathcal{P}|\right) = \exp\left(|\mathcal{P}| \left(\frac{\log(er)}{r} + \log n - n\right)\right),
\]
which tends to 0 as \( r \) goes to infinity (the second factor in the exponent becomes negative). Consequently, almost every \( H \) has a copy of \( G \) in each subset of \( \frac{1}{r}|V(H)| \) vertices.

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