On induced Folkman numbers

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Abstract

In 1970, Folkman proved that for any graph $G$ there exists a graph $H$ with the same clique number as $G$. In addition, any $r$-coloring of the vertices of $H$ yields a monochromatic copy of $G$. For a given graph $G$ and a number of colors $r$ let $f(G,r)$ be the order of the smallest graph $H$ with the above properties. In this paper, we give a relatively small upper bound on $f(G,r)$ as a function of the order of $G$ and its clique number.

1 Introduction

We write $H \rightarrow (G)_{v}^{r}$ if for every $r$-coloring of the vertices of $H$, there exists a monochromatic copy of $G$. If such a monochromatic copy is also an induced copy, then we write $H \rightarrow_{ind} (G)_{v}^{r}$. Let $\omega(G)$ be the clique number of $G$, i.e., the order of a maximal clique in $G$. Folkman [3] proved that for every graph $G$ there exists a graph $H$ such that $H \rightarrow (G)_{v}^{r}$ and $\omega(H) = \omega(G)$. Clearly $\omega(H) \geq \omega(G)$ for any graph with $H \rightarrow (G)_{v}^{r}$ and thus Folkman’s theorem is in this sense, the best possible. In this paper we consider a more general problem. Let the induced Folkman number be defined as

$$F(G,r) = \min\{|V(H)| : H \rightarrow_{ind} (G)_{v}^{r} \text{ and } \omega(H) = \omega(G)\}.$$  

Clearly, $F(G,r) \geq f(G,r)$, where $f(G,r)$ is the function considered in the abstract, and thus, an upper bound on $F(G,r)$ also yields an upper bound on $f(G,r)$. It was

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observed in [5] that $F(G, r)$ is well-defined for any graph $G$ and positive integer $r$. However, the argument from [5] does not give a good bound on $F(G, r)$. Subsequently, in [1], the authors gave a different proof which yields a relatively small bound.

**Theorem 1 ([1])** Let $r$ be a fixed natural number. Then, for every graph $G$ of order $n$,

$$F(G, r) \leq Cn^3 \log^3 n,$$

where $C > 0$ is a constant depending only on $r$.

It was also shown in [1] that $F(K_n, r) \leq Cn^2 \log^4 n$ for some constant $C = C(r)$.

In this paper, we extend previous results and refine the bound on $F(G, r)$, conditioning on the clique number of $G$.

**Theorem 2** Let $r$ be a fixed natural number. Then, for every graph $G$ of order $n$ and clique number $\omega$,

$$F(G, r) \leq Cn^3 \log^\omega n,$$

where $C > 0$ is a constant depending only on $r$ and $c > 0$ is an absolute constant.

**Remark 3** We prove Theorem 2 with $c = 5$ and in order to simplify the presentation we do not attempt to find the optimal constant $c$ which may be obtained using the approach we consider here. We also note that the best lower bound we know (attained for $\omega = \Theta(n)$) is quadratic in $n = |V(G)|$.

Finally, it is also worth mentioning that a related result (without any conditions on the clique number) was obtained by Eaton and the third author [2]. They showed that for every natural number $r$ there is a constant $C = C(r)$ such that for every graph $G$ of order $n$,

$$\min\{|V(H)| : H \rightarrow (G)_r\} \leq Cn^2 \log n.$$

The base of all logarithms in this paper is $e$.

## 2 Proof of Theorem 2

Set $c = 5$. We will assume that $n$ is a sufficiently large number, wherever necessary.

### 2.1 Construction

Let $G = (V, E)$ be a graph of order $n$ with $\omega = \omega(G)$. Note that if $\omega \leq \log^2 n$ then

$$\frac{n^3}{\omega} \log^c n \geq n^3 \log^{c-2} n = n^3 \log^3 n,$$
and consequently Theorem 1 yields the statement. Thus, we may assume that
\[ \log^2 n \leq \omega \leq n. \]

Let \( r \geq 1 \) be a fixed natural number. We fix a prime number \( q \) such that
\[ \sqrt{\frac{n^3}{\omega}} \log^{\frac{2}{3}} n \leq q + 1 \leq 2\sqrt{\frac{n^3}{\omega}} \log^{\frac{2}{3}} n, \tag{1} \]
which is always possible by Bertrand’s Postulate.

Let \( \mathcal{H} = (W, \mathcal{F}) \) be the \((q + 1)\)-uniform, \((q + 1)\)-regular hypergraph formed by the lines of the projective plane of order \( q \). It is well known that projective planes of order \( q \) exist for all primes \( q \) (see, e.g., [4]). Then \( \mathcal{H} \) has the following properties:

(a) For every pair of distinct vertices \( w, w' \in W \), there exists a unique hyperedge \( K \in \mathcal{F} \) such that \( w, w' \in K \);

(b) Moreover,
\[ |W| = |\mathcal{F}| = q^2 + q + 1 < (q + 1)^2 \leq 4\frac{n^3}{\omega} \log^c n. \tag{2} \]

We now construct a “random” graph \( H = H_G \) on \( \mathcal{H} \). We partition each hyperedge of \( \mathcal{H} \) into \( n + 1 \) sets of vertices where \( n \) of them have size
\[ x = \lfloor 2r \log n \rfloor, \]
i.e., for each hyperedge \( F \in \mathcal{F} \), we partition it into \( F = X_0 \cup X_1 \cup \cdots \cup X_n \) with \( |X_0| = q + 1 - nx \) and \( |X_i| = x \) for \( i = 1, 2, \ldots, n \). Note that
\[ |X_0| \geq \sqrt{\frac{n^3}{\omega}} \log^{\frac{2}{3}} n - 2rn \log n \geq n \log^{\frac{2}{3}} n - 2rn \log n \gg \log n, \]
and hence \( X_0 \) is much bigger than the other parts in the partition. For each hyperedge \( F \in \mathcal{F} \), we choose one such ordered partition uniformly at random. Suppose \( V = \{v_1, v_2, \ldots, v_n\} \) is the vertex set of \( G \). Then for each \( u \in X_i, w \in X_j, 1 \leq i < j \leq n \), we let \( \{u, w\} \) be an edge in \( H \) if and only if \( \{v_i, v_j\} \) is an edge in \( G \). Observe that the graph \( H \) obtained this way is well-defined because every pair of vertices in \( W \) is contained in exactly one hyperedge of \( \mathcal{H} \).

We will show that there exists a graph \( H \) constructed on \( \mathcal{H} \) with the following properties:

(i) \( \omega(H) = \omega(G) \);

(ii) For every set \( U \subseteq W, |U| = \lfloor |W| / r \rfloor \), the graph \( H[U] \) contains \( G \) as an induced subgraph.
2.2 Property (i)

To establish that $\omega(H) = \omega(G)$, we find it convenient to first introduce a randomly chosen graph $H' = H_{K_n}$, constructed similarly as $H_G$ but with $G$ replaced by $K_n$. In other words, every hyperedge $F \in \mathcal{F}$ contains a complete $n$-partite graph on $X_1 \cup \cdots \cup X_n$ and an empty graph on $X_0$. We show that with probability tending to one (as $n$ goes to infinity), the only copies of $K_{\omega+1}$ in $H'$ are those that are contained completely within some hyperedge of $\mathcal{H}$. This will prove that the graph $H$ obtained from $H'$ by deleting the extra edges cannot have a clique of size greater than that of $G$, and hence $\omega(H) = \omega(G)$.

We first derive an expression for the probability that any set of vertices induces a clique in $H'$. For any $S \subseteq W$, define

$$\mathcal{F}(S) = \{F \in \mathcal{F} : |S \cap F| \geq 2\}.$$  \hfill (3)

Let $A_S$ be the event that $S$ induces a clique in $H'$. Notice that if $A_S$ occurs, then the event $A_{S \cap F}$ occurs for every $F \in \mathcal{F}(S)$. Since all $A_{S \cap F}$ are independent, we have

$$\Pr(A_S) = \prod_{F \in \mathcal{F}(S)} \Pr(A_{S \cap F}).$$

Also notice that for $F \in \mathcal{F}(S)$, the probability $\Pr(A_{S \cap F})$ equals the probability that for a fixed partition $F = X_0 \cup \cdots \cup X_n$ a randomly selected subset $S_F \in \binom{S}{|S \cap F|}$ satisfies $S_F \cap X_0 = \emptyset$ and $|S_F \cap X_i| \leq 1$ for each $1 \leq i \leq n$. Thus,

$$\Pr(A_{S \cap F}) = \binom{n}{|S \cap F|} \left(\frac{q+1}{|S \cap F|}\right)^{|S \cap F|} \leq \left(\frac{n}{q+1}\right)^{|S \cap F|}.$$  \hfill (4)

Subsequently, the choice of $x$ yields

$$\Pr(A_S) \leq \left(\frac{n}{q+1}\right)^{\sum_{F \in \mathcal{F}(S)} |S \cap F|} \leq \left(\frac{2rn \log n}{q+1}\right)^{\sum_{F \in \mathcal{F}(S)} |S \cap F|}.$$  \hfill (4)

Let

$$\alpha = \frac{10 \log n}{3 \log \log n}.$$  \hfill (5)

Let $S \subseteq W$ be a subset of size $\omega + 1$ such that

$$S \not\subset F \quad \text{for all} \quad F \in \mathcal{F}. \hfill (5)$$

Observe that $\mathcal{F}(S)$ introduced in (3) satisfies $|\mathcal{F}(S)| \geq 2$. We consider the following two cases:

(I) $S$ intersects “many” hyperedges, which we quantitatively characterize by $\sum_{F \in \mathcal{F}(S)} |S \cap F| > \alpha(\omega + 1)$;
(II) $S$ intersects only a “few” hyperedges, which we quantitatively characterize by

$$\sum_{F \in \mathcal{F}(S)} |S \cap F| \leq \alpha (\omega + 1).$$

Let $A_1$ and $A_2$ denote the events that $H'$ contains a clique $H'[S] = K_{\omega+1}$ induced on a subset $S$ of types (I) and (II), respectively. Since the probability that $H'$ contains $K_{\omega+1}$ such that $K_{\omega+1} \not\subseteq F$ for all $F \in \mathcal{F}$ can be bounded from above by $\Pr(A_1) + \Pr(A_2)$, it suffices to show that each of these probabilities is $o(1)$ with respect to $n$.

**Claim 1** $\Pr(A_1) = o(1)$.

**Proof.** Let $S \subseteq W$ be a set of size $\omega + 1$ of type (I). From (4), the probability that $H'[S] = K_{\omega+1}$ is at most $\left(\frac{2r \log n}{q+1}\right)^{\alpha(\omega+1)}$. Thus,

$$\Pr(A_1) \leq \left(\frac{|W|}{\omega + 1}\right) \left(\frac{2rn \log n}{q + 1}\right)^{\alpha(\omega+1)} \leq \left(\left|W\right| \left(\frac{2rn \log n}{q + 1}\right)^{\alpha}\right)^{\omega+1}.$$

Since in view of (2) and (1), $|W| \leq n^4$ and $q + 1 \geq n \log^{\frac{c}{2}} n$,

$$\Pr(A_1) \leq \left(n^4 \left(\frac{2r}{\log^{\frac{c}{2}} n}\right)^{\alpha}\right)^{\omega+1} = \exp\left\{ (\omega + 1) \left(4 \log n - \left(\frac{c}{2} - 1\right) \alpha \log \log n + O(\alpha)\right) \right\}.$$

Finally from the choice of $c$ and $\alpha$, we obtain that

$$\Pr(A_1) \leq \exp\{ (\omega + 1)(- \log n + o(\log n))\} = o(1).$$

□

**Claim 2** $\Pr(A_2) = o(1)$.

**Proof.** Let $S \subseteq W$ be a set of type (II). For every $w \in S$, let

$$\deg(w) = |\{F : F \in \mathcal{F}(S) \text{ and } w \in F\}|.$$

Observe that $\sum_{w \in S} \deg(w) = \sum_{F \in \mathcal{F}(S)} |S \cap F| \leq \alpha(\omega + 1)$. This implies that there exists some $w \in S$ such that $\deg(w) \leq \alpha$. Also, since every pair of vertices in $S$ belongs to some $F \in \mathcal{F}(S)$, we conclude that there exists a hyperedge $F_0 \in \mathcal{F}(S)$ with $w \in F_0$ such that $|S \cap F_0| \geq \frac{\omega}{\alpha}$. Moreover, since (5) implies that $S \not\subseteq F_0$, there exists at least one vertex $u \not\in S \setminus F_0$. Now set

$$t = \left\lceil \frac{\omega}{\alpha} \right\rceil.$$

Consequently, every set $S$ of type (II) contains a subset $T$ of size $t + 1$ inducing a clique $K_{t+1}$ in which precisely $t$ vertices lie in some hyperedge of $\mathcal{H}$.  

5
Let $A_3$ be the event that there is a set $T$ of size $t + 1$ inducing a clique $K_{t+1}$ in which precisely $t$ vertices lie in some hyperedge of $\mathcal{H}$. Clearly, $\Pr(A_2) \leq \Pr(A_3)$ and thus it suffices to show that $\Pr(A_3) = o(1)$. First note that

$$\sum_{F \in \mathcal{F}(T)} |T \cap F| = t + 2t = 3t.$$ 

Consequently, by (4)

$$\Pr(A_3) \leq |\mathcal{F}| \binom{q + 1}{t} |W| \left(\frac{2rn \log n}{q + 1}\right)^{3t}.$$ 

Since $|W| = |\mathcal{F}| \leq (q + 1)^2$,

$$\Pr(A_3) \leq (q + 1)^4 \left(\frac{\log n}{q + 1}\right)^t \left(\frac{8r^3n^3 \log^3 n}{(q + 1)^3}\right)^t \leq (q + 1)^4 \left(\frac{8r^3n^3 \log^3 n}{t(q + 1)^2}\right)^t.$$ 

Recall that $t \geq \frac{c \omega}{\alpha}$ and $(q + 1)^4 \geq \frac{n^3}{\omega} \log^c n$. Thus,

$$\Pr(A_3) \leq (q + 1)^4 \left(\frac{8r^3 \alpha}{\log^c n}\right)^t.$$ 

Moreover, since trivially, $(q + 1)^4 \leq n^7$ and $\alpha \leq \log n$, we obtain that

$$\Pr(A_3) \leq n^7 \left(\frac{8er^3}{\log^c n}\right)^t = \exp\{7 \log n - (c - 4)t \log \log n + O(t)\}.$$ 

Recall that by assumption $\omega \geq \log^2 n$. Hence,

$$t \geq \frac{\omega}{\alpha} \geq \log^2 n \frac{3 \log \log n}{10 \log n} \geq \log n,$$

and consequently, $\Pr(A_3)$ goes to zero as $n$ tends to infinity. This completes the proof of Claim 2 and Property (i).

2.3 Property (ii)

Let $\mathcal{H} = (W, \mathcal{F})$ be the hypergraph defined in 2.1. For a fixed $U \subseteq W$, with $|U| = |W|/r$ (for simplicity, we assume that this is an integer), let $B_U$ denote the event that $H[U]$ contains no induced copy of $G$. Note that if $B_U$ occurs, then all events $B_{U \cap F}$, $F \in \mathcal{F}$ must occur. Thus, $B_U \subseteq \bigcap_{F \in \mathcal{F}} B_{U \cap F}$ and since all events $B_{U \cap F}$ are independent,

$$\Pr(B_U) \leq \prod_{F \in \mathcal{F}} \Pr(B_{U \cap F}).$$ (6)
We now obtain an upper bound on $\Pr(B_{U \cap F})$ for a fixed hyperedge $F \in \mathcal{F}$. Let $|U \cap F| = u_F$. Note that for a fixed $i$, $1 \leq i \leq n$, the probability that $U \cap X_i = \emptyset$ is equal to the probability that for a fixed partition $F = X_0 \cup \cdots \cup X_n$, a randomly chosen subset $T$ with $|T| = u_F$ satisfies $T \cap X_i = \emptyset$. Hence, in view of $|X_i| = x$ for $1 \leq i \leq n$,

$$\Pr(B_{U \cap F}) \leq n \left( \frac{q + 1 - x}{u_F} \right) \leq n \exp \left\{ -\frac{x u_F}{q + 1} \right\}.$$

Consequently, by (6) and the fact that $|F| = |W|$,

$$\Pr(B_U) \leq n^{|W|} \exp \left\{ -\frac{x}{q + 1} \sum_{F \in \mathcal{F}} u_F \right\}.$$

Since $\mathcal{H}$ is $(q + 1)$-regular, for any $U \subseteq W$,

$$\sum_{F \in \mathcal{F}} u_F = \sum_{F \in \mathcal{F}} |U \cap F| = |U|(q + 1) = (q + 1)|W|/r.$$

Thus,

$$\Pr(B_U) \leq n^{|W|} \exp \left\{ -x |W|/r \right\}.$$

We can bound the probability that there exists some $U \subseteq W$ with $|U| = |W|/r$ such that the graph induced by $H[U]$ does not induce a copy of $G$ by bounding the probability of the union of the events $B_U$ over all subsets $U \subseteq W$. Thereby, we obtain

$$\Pr \left( \bigcup_U B_U \right) \leq \left( \frac{|W|}{|W|/r} \right) n^{|W|} \exp \left\{ -x |W|/r \right\}$$

$$\leq (er)^{|W|/r} n^{|W|} \exp \left\{ -x |W|/r \right\}$$

$$= \exp \left\{ |W|/r \left( 1 + \log r + r \log n - x \right) \right\}.$$

Since $x = \lfloor 2r \log n \rfloor > 2r \log n - 1$,

$$1 + \log r + r \log n - x \leq 2 + \log r - r \log n \ll 0$$

for $n$ sufficiently large. Thus, $\Pr(\bigcup_U B_U) = o(1)$ and so $\Pr(\bigcap_U \overline{B_U}) = 1 - o(1)$, i.e., almost every graph $H$ constructed on $\mathcal{H}$ satisfies $H \rightarrow (G)_{r,v}$. This completes the proof of Theorem 2.

### 3 Remarks on the lower bound

Let $G$ be a graph of order $n$ with clique number $\omega$. In this note, we gave a relatively small upper bound on $F(G, r)$ as a function of $n$ and $\omega$. Here we observe a simple lower bound on $F(G, r)$.
Consider the graph $G_0 = K_\omega \cup \overline{K}_{n-\omega}$ and let $H$ be the graph satisfying $H \xrightarrow{\text{ind}} (G_0)_2^\nu$.

Clearly, $H$ must contain many copies of $K_\omega$. Starting with $H$, we find a copy of $K_\omega$ and remove it obtaining a graph $H_1$ (of order $|V(H)| - \omega$). We continue the process of removing $K_\omega$ repeatedly until we can no longer find a copy of $K_\omega$. Let $\ell$ denote the number of repetitions and $H_\ell$ be the $K_\omega$-free graph eventually obtained. We color the vertices of $H_\ell$ blue and the vertices of $H \setminus H_\ell$ red. Since $H_\ell$ does not contain any copy of $K_\omega$, it also contains no copy of $G_0$. Since $H \xrightarrow{\text{ind}} (G_0)_2^\nu$, $H \setminus H_\ell$ must contain $G_0$.

On the other hand, the vertex set of $H \setminus H_\ell$ is a union of $\ell$ vertex disjoint cliques, and hence, it contains no independent set of size bigger than $\ell$. Therefore, in order for $G_0 = K_\omega \cup \overline{K}_{n-\omega}$ to be an induced subgraph of $H \setminus H_\ell$, we must have $\ell \geq n - \omega$.

Consequently, $|V(H)| \geq |V(H \setminus H_\ell)| \geq \ell \omega \geq (n - \omega)\omega$.

Therefore,
\[
\max_G \{F(G, 2)\} \geq (n - \omega)\omega,
\]

where the maximum is taken over all graphs $G$ of order $n$. In particular, by Theorem 2 we get
\[
\max_G \{F(G, 2)\} = \Theta \left( n^{2+o(1)} \right),
\]

where the maximum is taken over all graphs $G$ of order $n$ with clique number equals $n/2$.

In view of the current upper bound on $F(G, r)$ we propose the following problem.

**Problem 1** Prove or disprove that
\[
\lim_{n \to \infty} \max_G \frac{F(G, r)}{n^2} = \infty,
\]

where the maximum is taken over all graphs $G$ of order $n$ and $r$ is some fixed natural number.

Notice that in order to give an affirmative answer to Problem 1 it is enough to find a family of graphs $\mathcal{G} = \{G_1, G_2, \ldots \}$, $G_n$ of order $n$, $n \geq 1$, such that
\[
\lim_{n \to \infty} \frac{F(G_n, r)}{n^2} = \infty.
\]

Let $\mathcal{G}_0$ be a family of graphs defined in the previous paragraphs with $\omega = \frac{n}{2}$, i.e., $G_0 = K_{\frac{n}{2}} \cup \overline{K}_{\frac{n}{2}}$. We already showed that $F(G_0, r) = \Omega(n^2)$. It is not clear if $F(G_0, r) \gg n^2$.

However, it is easy to see that without any conditions on the clique number we get
\[
\min \{|V(H)| : H \xrightarrow{\text{ind}} (G_0)_2^\nu \} = O(n^2).
\]

Indeed, let $H$ be a union of $\frac{rn}{2} + 1$ vertex disjoint cliques $K_{r(\frac{n}{2}-1)+1}$. Then by the pigeonhole principle any $r$-coloring of vertices of $H$ yields a monochromatic union of $\frac{rn}{2} + 1$ vertex disjoint cliques $K_{\frac{n}{2}}$. Consequently, $H \xrightarrow{\text{ind}} (G_0)_2^\nu$ and $|V(H)| = O(n^2)$, as required.
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References


