5. Random Processes

5-1 Introduction
5-2 Continuous and Discrete Random Processes
5-3 Deterministic and Nondeterministic Random Processes
5-4 Stationary and Nonstationary Random Processes
5-5 Ergodic and Nonergodic Random Processes
5-6 Measurement of Process Parameters
5-7 Smoothing Data with a Moving Window Average

Topics

- Continuous and Discrete Random Processes
- Classification of Random Processes
- Deterministic and Nondeterministic Random Processes
- Stationary and Nonstationary Random Processes
- Wide Sense Stationary
  - Time Averages and Statistical Mean
  - Autocorrelation early intro
- Ergodic and Non-ergodic Random Processes
- Ergodicity
- Measurement of Process Parameters
  - Statistics
- Smoothing Data with a Moving Window Average
- Simulating a Random Process
Chapter 5: Random Processes

A random process is a collection of time functions and an associated probability description.

When a continuous or discrete or mixed process in time/space can be describe mathematically as a function containing one or more random variables.

- A sinusoidal waveform with a random amplitude.
- A sinusoidal waveform with a random phase.
- A sequence of digital symbols, each taking on a random value for a defined time period (e.g. amplitude, phase, frequency).
- A random walk (2-D or 3-D movement of a particle)

The entire collection of possible time functions is an ensemble, designated as \( \{x(t)\} \), where one particular member of the ensemble, designated as \( x(t) \), is a sample function of the ensemble. In general only one sample function of a random process can be observed!

Think of:

\[
X(t) = A \cdot \sin(w \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi
\]

where \( A \) and \( w \) are known constants.

Note that once a sample has been observed …

\[
x_1(t_1) = A \cdot \sin(w \cdot t_1 + \theta)
\]

the function is known for all time, \( t \).

Note that, \( x(t_2) \) is a second time sample of the same random process and does not provide any “new information” about the value of the random variable.

\[
x_1(t_2) = A \cdot \sin(w \cdot t_2 + \theta)
\]

There are many similar ensembles in engineering, where the sample function, once known, provides a continuing solution. In many cases, an entire system design approach is based on either assuming that randomness remains or is removed once actual measurements are taken!

For example, in communications there is a significant difference between coherent (phase and frequency) demodulation and non-coherent (i.e. unknown starting phase) demodulation.

On the other hand, another measurement in a different environment might measure

\[
x_2(t_1) = A_2 \cdot \sin(w \cdot t_1 + \theta_1)
\]

In this “space” the random variables could take on other values within the defined ranges. Thus an entire “ensemble” of possibilities may exist based on the random variables defined in the random process.
For example, assume that there is a known AM signal transmitted:

\[ s(t) = (1 + b \cdot A(t)) \cdot \sin(w \cdot t) \]

at an undetermined distance the signal is received as

\[ y(t) = (1 + b \cdot A(t)) \cdot \sin(w \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi \]

The received signal is mixed and low pass filtered …

\[ x(t) = h(t) \otimes [y(t) \cdot \cos(w \cdot t)] = h(t) \otimes [(1 + b \cdot A(t)) \cdot \sin(w \cdot t + \theta) \cdot \cos(w \cdot t)] 0 \leq \theta \leq 2 \cdot \pi \]

\[ x(t) = h(t) \otimes [y(t) \cdot \cos(w \cdot t)] = h(t) \otimes [(1 + b \cdot A(t)) \cdot 0.5 \cdot [\sin(2 \cdot w \cdot t + \theta) + \sin(\theta)]], 0 \leq \theta \leq 2 \cdot \pi \]

If the filter removes the 2wt term, we have

\[ x(t) = h(t) \otimes [y(t) \cdot \cos(w \cdot t)] = \frac{(1 + b \cdot A(t))}{2} \cdot \sin(\theta), \quad 0 \leq \theta \leq 2 \cdot \pi \]

Notice that based on the value of the random variable, the output can change significantly! From producing no output signal, \((\theta = 0, \pi)\), to having the output be positive or negative \((\theta = 0 to \pi or \pi to 2\pi)\). P.S. This is not how you perform non-coherent AM demodulation.

To perform coherent AM demodulation, all I need to do is measured the value of the random variable and use it to insure that the output is a maximum (i.e. mix with \(\cos(w \cdot t + \theta_m)\), where \(\theta_m = \theta(t)\))

Note: the phase is a function of frequency, time, and distance from the transmitter.

Terminology to learn:

- Continuous vs. discrete
- Deterministic vs. nondeterministic
- Stationary vs. nonstationary
- Ergodic vs. nonergodic
“Degrees of freedom” Exercise 5-1.1

a) If it is assumed that any random process can be described by picking one descriptor from each of the four pairs defined, how many “classes” of random processes can be described.

- Continuous vs. discrete
- Deterministic vs. nondeterministic
- Stationary vs. nonstationary
- Ergodic vs. nonergodic

4 things that one or the other is selected. Therefore,

\[ \text{Descriptors} = 2^4 = 16 \]

b) If you considered mixed processes in which two are combined (from the classes defined above), how many possible class description of the mixed process can be defined.

2 random processes each with 16 classes. Therefore

\[ \text{Descriptors} = 16^2 = 256 \]

Things get messy way too fast …
Continuous and Discrete Random Processes

A continuous random process is one in which the random variables, such as \( X(t_1), X(t_2), \ldots X(t_n) \), can assume any value within the specified range of possible values. A more precise definition for a continuous random process also requires that the probability distribution function be continuous.

Examples are:

- Thermal agitation noise in a conductor,
- Shot noise in electronic tubes or transistors,
- Wind velocity
- As an exception, uniform distributions are usually thought of as continuous.

A discrete random process is one in which the random variables, such as \( X(t_1), X(t_2), \ldots X(t_n) \), can assume any certain values (though possibly an infinite number of values). A more precise definition for a discrete random process also requires that the probability distribution function consist of numerous discontinuities or steps. Alternately, the probability density function is better defined as a probability mass function … the pdf is composed of delta functions.

Examples are:

- ADC outputs
- Truncated or rounded numbers

A mixed random process consists of both continuous and discrete components. The probability distribution function consists of both continuous regions and steps. The pdf has both continuous regions and delta functions.

Examples are:

- Half-wave or full-wave rectifiers
- Discrete time samples (a sample and hold device)
- Analog limiters (saturated or limited Op-Amp outputs)

Note that all of the above signals may be considered continuous in time or may be sampled at any arbitrary time.
Exercise 5-2.2

A random time function has a mean value of 1 and amplitude that has an exponential distribution. This function is multiplied by a sinusoid of unit amplitude and phase uniformly distributed over (0, 2π).

a) Classify the product as continuous, discrete or mixed.
   - continuous amplitude distribution and a continuous time signal with a continuous (uniform) distribution

b) After passing the above signal through an ideal hard limiter, is the output continuous, discrete or mixed.
   - The signal has been discretized in value; therefore, one would expect a pmf description of the signal behavior.

c) If the sinusoid is passed through a half-wave rectifier before multiplying the exponentially distributed time function, is the output continuous, discrete or mixed.
   - The flat spot (due to half-wave rectification) defines a pmf level among the continuous outputs possible, making this a mixed signal.
Problem 5-2.1

Classify each of the following random processes as continuous, discrete or mixed.

a) A random process in which the random variable is the number of cars per minute passing a traffic counter.

b) The thermal noise voltage generated by a resistor.

c) The random process defined in problem 5-1.2. (Discrete sample addition)

d) The random process that results when a Gaussian random process is passed through an ideal half-wave rectifier.

e) The random process that results when a Gaussian random process is passed through an ideal full-wave rectifier.

f) A random process having sample functions of the form:

\[ X(t) = A \cdot \cos(B \cdot t + \theta) \]

where A is a constant, B is a R.V. exponentially distributed from 0 to inf, and theta is uniformly distributed from 0 to 2 pi.

Discrete: (a) and (c)

Continuous: (b), (e), and (f)

Mixed: (d)

A strong hint: Is the density function continuous or discrete (delta function components) or both?
**Deterministic and Nondeterministic Random Processes**

A nondeterministic random process is one where future values of the ensemble cannot be predicted from previously observed values (a sequence of random variables versus one random variable in the model equation).

Many of the random processes that you will deal with are nondeterministic, every sample function measured must be described based on it’s probability distribution and density.

Example include:
- a process composed of independent, identically distributed random sample
- Brownian motion
- Random walk
- Typical definition of noise

A deterministic random process is one where one or more observed samples allow all future values of the sample function to be predicted (or pre-determined). For these processes, a single random variable may exist for the entire ensemble. Once it is determined (one or more measurements) the sample function is known for all t.

Examples include:
- Radio station carrier frequency and phase values at a location
- Fourier coefficients of a measured signal
  \[ X(t) = \sum_{n=0}^{\infty} A_n \cdot \cos(2\pi \cdot n \cdot f_0 t) + B_n \cdot \sin(2\pi \cdot n \cdot f_0 t) \]
  where both \( A_n \) and \( B_n \) are independent random variables that are fixed for any particular sample function of the possible ensemble.
- An exponential decay
  \[ X(t) = A \cdot \exp(-\beta \cdot t), \text{ for } t \geq 0 \]
  where both \( A \) and \( \beta \) are independent random variables
- The flight path of a baseball or football
Problem 5-3.1

State whether each of the random processes described in problem 5-2.1 is deterministic or nondeterministic.

a) A random process in which the random variable is the number of cars per minute passing a given traffic counter.

b) The thermal noise voltage generated by a resistor.

c) The random process defined in problem 5-1.2. (Discrete sample addition)

d) The random process that results when a Gaussian random process is passed through an ideal half-wave rectifier.

e) The random process that results when a Gaussian random process is passed through an ideal full-wave rectifier.

f) A random process having sample functions of the form:

\[ X(t) = A \cdot \cos(B \cdot t + \theta) \]

where A is a constant, B is a R.V. exponentially distributed from 0 to inf, and theta is uniformly distributed from 0 to 2 pi.

Deterministic: (f)

Non-deterministic: (a), (b), (c), (d), (e)
Stationary and Nonstationary Random Processes

The probability density functions for random variables in time have been discussed, but what is the dependence of the density function on the value of time, $t$, when it is taken?

If all marginal and joint density functions of a process do not depend upon the choice of the time origin, the process is said to be stationary (that is it doesn’t change with time). All the mean values and moments are constants and not functions of time!

For nonstationary processes, the probability density functions change based on the time origin or in time. For these processes, the mean values and moments are functions of time.

In general, we always attempt to deal with stationary processes … or approximate stationary by assuming that the process probability distribution, means and moments do not change significantly during the period of interest.

Examples:

- Resistor values (noise varies based on the local temperature)
- Wind velocity (varies significantly from day to day)
- Humidity (though it can change rapidly during showers)

The requirement that all marginal and joint density functions be independent of the choice of time origin is frequently more stringent (tighter) than is necessary for system analysis. A more relaxed requirement is called stationary in the wide sense: where the mean value of any random variable is independent of the choice of time, $t$, and that the correlation of two random variables depends only upon the time difference between them. That is

$$E[X(t)] = \overline{X} = \mu_X \quad \text{and}$$

$$E[X(t_1) \cdot X(t_2)] = E[X(0) \cdot X(t_2 - t_1)] = \overline{X}'(0) \cdot \overline{X}'(\tau) = R_{XX}(\tau) \quad \text{for} \quad \tau = t_2 - t_1$$

You will typically deal with **Wide-Sense Stationary Signals**.

… the autocorrelation function is a key ….

Note: if a process is slowly timing varying, measurements are used to “estimate” the current mean and variance so that we can assume these are correct over the measurement period.

- This creates a class of “adaptive systems” that can follow slow variations in time and allow operation to continue.
Chapter 6 Information early: The Autocorrelation Function

For a sample function defined by samples in time of a random process, how alike are the different samples?

Define: \( X_1 = X(t_1) \) and \( X_2 = X(t_2) \)

The autocorrelation is defined as:

\[
R_{xx}(t_1, t_2) = E[X_1 X_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2) \, dx_1 \, dx_2
\]

The above function is valid for all processes, stationary and non-stationary.

For WSS processes:

\[
R_{xx}(t_1, t_2) = E[X(t)X(t+\tau)] = R_{xx}(\tau)
\]

For non-WSS processes:

\[
R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)] = R_{xx}(t_1, t_2)
\]

If the process is ergodic, the time average is equivalent to the probabilistic expectation, or

\[
\mathbb{R}_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t+\tau) \, dt = \langle x(t) \cdot x(t+\tau) \rangle
\]

and

\[
\mathbb{R}_{xx}(\tau) = R_{xx}(\tau)
\]

As a note for things you’ve been computing,

\[
R_{xx}(t_1, t_1) = R_{xx}(0) = E[X_1 X_1] = E[X_1^2] = \int_{-\infty}^{\infty} x_1^2 \cdot f(x_1) \, dx_1 = \sigma_x^2 + \mu_x^2
\]

\[
\mathbb{R}_{xx}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)^2 \, dt = \langle x(t)^2 \rangle
\]
Problem 5-4.1

State whether each of the random processes described in problem 5-2.1 may reasonably be considered stationary or nonstationary. If you describe a process as nonstationary, state the reason for this claim.

a) A random process in which the random variable is the number of cars per minute passing a given traffic counter.

b) The thermal noise voltage generated by a resistor.

c) The random process defined in problem 5-1.2. (Discrete sample addition)

d) The random process that results when a Gaussian random process is passed through an ideal half-wave rectifier.

e) The random process that results when a Gaussian random process is passed through an ideal full-wave rectifier.

f) A random process having sample functions of the form:

\[ X(t) = A \cdot \cos(B \cdot t + \theta) \]

where A is a constant, B is a R.V. exponentially distributed from 0 to inf, and theta is uniformly distributed from 0 to 2 pi.

Stationary: (b), (d), (e), (f)

Non-Stationary: (a) traffic changes based on time of day, (c) depends upon time from 0-9, non-stationary, then beyond 9 it becomes stationary.
Ergodic and Nonergodic Random Processes

Ergodicity deals with the problem of determining the statistics of an ensemble based on measurements from a sample function of the ensemble.

For ergodic processes, all the statistics can be determined from a single function of the process. This may also be stated based on the time averages. For an ergodic process, the time averages (expected values) equal the ensemble averages (expected values). That is to say,

$$
X^n = \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) \cdot dt
$$
or better stated as

$$
E[X^n] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) \cdot dt
$$

Note that ergodicity cannot exist unless the process is stationary!

A nonergodic process is one where any or all of these properties do not exist. All nonstationary processes are nonergodic.

An example of a stationary process that is not ergodic is.

$$
X(t) = Y \cdot \sin(w \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi \text{ and } Y \text{ a r.v.}
$$

For this random process, the ensemble consists of two random variable, phase and amplitude. For this process, Y takes on different values for different sample functions, while all processes take on the same random phase once it is known.

While it is exceedingly difficult to prove ergodicity, it is customary to assume ergodicity for most of the problems that we will deal with (unless there is an obvious reason not to, like non-stationary).
Problem 5-5.1

State whether each of the random processes described in problem 5-2.1 may reasonably be considered ergodic or non-ergodic.

a) A random process in which the random variable is the number of cars per minute passing a given traffic counter.

b) The thermal noise voltage generated by a resistor.

c) The random process that results when a Gaussian random process is passed through an ideal half-wave rectifier.

d) A random process having sample functions of the form:

$$X(t) = A \cdot \cos(B \cdot t + \theta)$$

where $A$ is a constant, $B$ is a R.V. exponentially distributed from 0 to inf, and $\theta$ is uniformly distributed from 0 to 2 pi.

Ergodic: (b), (c)

Non- Ergodic: (a) traffic changes (not stationary) and (d) significant differences in ensemble members, therefore, time based statistics vary with $A$. 

Measurement of Process Parameters

For a stationary process, the statistical parameters of a random process are derived from measurements of the random variables $X(t)$ at multiple time instances, $t$.

As might be anticipated, the statistics may be generated based on one sample function. Therefore, it is not possible to generate an ensemble average. If the process is ergodic, the time average is equivalent to the ensemble average. As might be expected, ergodicity is typically assumed.

$$
\overline{X^n} = \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) \cdot dt
$$

Further, since an infinite time average is not possible, the statistical values (sample mean and variance) defined in Chapter 4 are used to estimate the appropriate moments. For a continuous time or discrete sample function, this becomes (simplified)

$$
\langle X \rangle = \overline{X} = \frac{1}{T} \int_{0}^{T} X(t) \cdot dt \quad \text{or} \quad \langle X \rangle = \overline{X} = \frac{1}{N} \cdot \sum_{n=1}^{N} X(k)
$$

$$
\langle X^2 \rangle = \overline{X^2} = \frac{1}{T} \int_{0}^{T} X(t)^2 \cdot dt \quad \text{or} \quad \langle X^2 \rangle = \overline{X^2} = \frac{1}{N} \cdot \sum_{n=1}^{N} X(k)^2
$$

etc …

Note: there are notation challenges here, while the bar hat notations worked for the time based average, it will not work for powers in $n$ … so the book and notes notations may be a problem!

As may be expected, we will be forming the time/sample statistical elements and comparing or computing the probabilistic elements to see if they are equivalent (ergodicity). In some cases biased statistics that may be able to be corrected to become equivalent will also be identified.
As before, for $X$ a random variable

\[
E[X] = E\left[ \frac{1}{T} \int_{0}^{T} X(t) \cdot dt \right] = \frac{1}{T} \int_{0}^{T} E[X(t)] \cdot dt
\]

The discrete version of this is a repeat of the sample mean …

Sample Mean

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

\[
E[\bar{X}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} \sum_{i=1}^{n} \bar{X} = \bar{X}
\]

For discrete samples, it is desired and, in fact, assumed that the observed samples are spaced far enough apart in time to be **statistically independent**.

The 2\(^{nd}\) moment needed to compute the variance of the estimated sample mean is computed as

\[
E[\bar{X}^2] = E\left[ \left( \frac{1}{n} \sum_{j=1}^{n} X_j \right) \cdot \left( \frac{1}{n} \sum_{j=1}^{n} X_j \right) \right] = E\left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} X_i \cdot X_j \right]
\]

For $X_i$ independent

\[
E[X_i \cdot X_j] = \begin{cases} E[X_i^2] = (\bar{X})^2 & \text{for } i = j \\ E[X_i] \cdot E[X_j] = (\bar{X})^2 & \text{for } i \neq j \end{cases}
\]

\[
E[\bar{X}^2] = \frac{1}{n^2} \sum_{i=1}^{n} \left\{ E[X_i \cdot X_i] + \sum_{j=1, j \neq i}^{n} E[X_i \cdot X_j] \right\}
\]

\[
E[\bar{X}^2] = \frac{1}{n^2} \left\{ n \cdot E[X^2] + (n^2 - n) E[X^2] \right\}
\]

\[
E[\bar{X}^2] = \frac{1}{n} \cdot (\bar{X})^2 + \frac{(n^2 - n)}{n^2} \cdot (\bar{X})^2 = \frac{1}{n} \left[ \left( \frac{\bar{X}^2}{n} \right) - (\bar{X})^2 \right] + \left( 1 - \frac{1}{n} \right) \cdot (\bar{X})^2 + \frac{1}{n} \cdot (\bar{X})^2
\]

\[
E[\bar{X}^2] = \frac{1}{n} \cdot \sigma_X^2 + (\bar{X})^2
\]
And the variance of the mean estimate is

$$\text{Var}[\bar{X}] = E[\bar{X}^2] - E[\bar{X}]^2 = \frac{1}{n} \cdot \sigma_X^2 + (\bar{X})^2 - \left(\frac{1}{n} \cdot \sigma_X^2\right)^2$$

As before, as the number of values goes to infinity, the sample mean becomes a better estimate of the actual mean.

As may be expected, the unbiased estimate of the sample variance can be defined as

$$\hat{\sigma}_X^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} X_i^2 - \frac{n}{n-1} \cdot (\hat{X})^2$$

**Exercise 5-6.2**

Show that the estimate of the variance is an unbiased estimate of the true variance.

$$E[\hat{\sigma}_X^2] = \sigma_X^2$$

$$E[\hat{\sigma}_X^2] = E\left[\frac{1}{n-1} \cdot \sum_{i=1}^{n} X_i^2 - \frac{n}{n-1} \cdot (\hat{X})^2\right]$$

$$E[\hat{\sigma}_X^2] = \frac{1}{n-1} \cdot \sum_{i=1}^{n} E[X_i^2] - \frac{n}{n-1} \cdot E[(\hat{X})^2]$$

$$E[\hat{\sigma}_X^2] = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left[\sigma_X^2 + \mu_X^2\right] - \frac{n}{n-1} \cdot \left[\frac{1}{n} \cdot \sigma_X^2 + (\bar{X})^2\right]$$

$$E[\hat{\sigma}_X^2] = \frac{n}{n-1} \cdot \sigma_X^2 + \frac{n}{n-1} \cdot \mu_X^2 - \frac{1}{n-1} \cdot \sigma_X^2 - \frac{n}{n-1} \cdot \mu_X^2$$

$$E[\hat{\sigma}_X^2] = \left[\frac{n}{n-1} - \frac{1}{n-1}\right] \cdot \sigma_X^2 + \left[\frac{n}{n-1} - \frac{n}{n-1}\right] \cdot \mu_X^2$$

$$E[\hat{\sigma}_X^2] = \sigma_X^2$$
A Process for Determining Stationarity and Ergodicity

a) Find the mean and the 2nd moment based on the probability

b) Find the time sample mean and time sample 2nd moment based on time averaging.

c) If the means or 2nd moments are functions of time … non-stationary

d) If the time average mean and moments are not equal to the probabilistic mean and moments or if it is not stationary, then it is non ergodic.

Examples:

$$\bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx$$
and

$$\bar{X^2} = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) \cdot dx$$

$$\langle x \rangle = \hat{X} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot dt$$
and

$$\langle x^2 \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [x(t)]^2 \cdot dt$$

Examples:

1) $x(t) = const.$
Deterministic, stationary, ergodic

2) $x(t) = A \cdot t$, for A a zero mean, unit variance Gaussian random variable.
Deterministic, stationary, non-ergodic

3) $x(t) = A \cdot \sin(wt)$, for A a uniformly distributed random variable $A \in [-2,2]$.
Deterministic, stationary in the mean but not the variance(!), non-ergodic

4) $x(t) = B_n \cdot rect\left(\frac{t-nT}{T}\right)$, for B = +/- 1 with prob. 50%.
Non-deterministic, Zero mean, stationary, ergodic (Bipolar/Binary communications)

5) $x(t) = A \cdot \cos(wt + \theta)$, for A and/or theta random variables ...
Deterministic, stationary in the mean but not the variance(!), non-ergodic