Chapter 6: Correlation Functions

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Concepts

- Autocorrelation
  - of a Binary/Bipolar Process
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- Linear Transformations
- Properties of Autocorrelation Functions
- Measurement of Autocorrelation Functions
- An Anthology of Signals
- Crosscorrelation
- Properties of Crosscorrelation
- Examples and Applications of Crosscorrelation
- Correlation Matrices For Sampled Functions
Chapter 6: Correlation Functions

We have already seen correlation functions a number of times in defining how similar one signal is to another or the signal to itself as a fixed sequence or random variables (Chap. 3 and 4).

From probability the correlation coefficient was defined: $-1 \leq \rho \leq 1$

$$\rho = \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y}$$

From sampled statistics Pearson’s $r$ was defined: $-1 \leq r \leq 1$

$$\rho = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sqrt{\left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right)\left(\sum_{i=1}^{n} (y_i - \bar{y})^2\right)}}$$

In defining a wide sense stationary process, the probabilistic expectation of a random process in time was defined in order to determine stationarity:

$$E[X(t_1)X(t_2)] = E[X(t)X(t + \tau)] = E[X(0)X(\tau)]$$

In fact for ergodic processes, the probabilistic operations have equivalent sample average statistics such that

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) \cdot dt$$

From this basis, we are interested in observing signal and systems that are stationary, ergodic, have sample statistics that can be readily collected and relate to know probability distributions. This allows the mathematical derivation of systems design concepts, the simulation have possible outcomes, and the collection of statistics on a prototype system to validate the theoretical performance!

So, on with additional properties and mathematical descriptions for wide-sense stationary (WSS), ergodic random processes.
Chapter 6: Correlation Functions

The Autocorrelation Function

For a sample function defined by samples in time of a random process, how alike are the different samples?

Define: \( X_1 = X(t_1) \) and \( X_2 = X(t_2) \)

The autocorrelation is defined as:

\[
R_{XX}(t_1, t_2) = E[X_1X_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdot \{x_1x_2, f(x_1, x_2)\}
\]

The above function is valid for all processes, stationary and non-stationary.

For WSS processes:

\[
R_{XX}(t_1, t_2) = E[X(t)X(t + \tau)] = R_{XX}(\tau)
\]

If the process is ergodic, the time average is equivalent to the probabilistic expectation, or

\[
\mathbb{R}_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t + \tau) \cdot dt = \langle x(t) \cdot x(t + \tau) \rangle
\]

and

\[
\mathbb{R}_{XX}(\tau) = R_{XX}(\tau)
\]

As a note for things you’ve been computing, the “zeroth lag of the autocorrelation” is

\[
R_{XX}(t_1, t_1) = R_{XX}(0) = E[X_1X_1] = E[X_1^2] = \int_{-\infty}^{\infty} dx_2 \cdot \{x_1^2, f(x_1)\} = \sigma_X^2 - \mu_X^2
\]

\[
\mathbb{R}_{XX}(0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{T} x(t)^2 \cdot dt = \langle x(t)^2 \rangle
\]
Demonstrating the similarity definition for the autocorrelation

Let \( Y(t) = X(t) - \rho \cdot X(t + \tau) \)

We wish to find the value of \( \rho \) that minimize the mean squared value of \( Y \).

Derivation:

\[
\min_{\rho} \left\{ E[Y(t)^2] \right\}
\]

\[
E[Y(t)^2] = E[(X(t) - \rho \cdot X(t + \tau))^2]
\]

\[
E[Y(t)^2] = E[X(t)^2 - 2 \cdot \rho \cdot X(t) \cdot X(t + \tau) + \rho^2 \cdot X(t + \tau)^2]
\]

Take the derivative with respect to \( \rho \), set the result equal to zero and determine the value.

\[
0 = E[-2 \cdot X(t) \cdot X(t + \tau) + 2 \cdot \rho \cdot X(t + \tau)^2]
\]

\[
E[X(t) \cdot X(t + \tau)] = \rho \cdot E[X(t + \tau)^2]
\]

\[
\rho \cdot E[X(t)^2] = R_{xx}(\tau)
\]

\[
\rho = \frac{R_{xx}(\tau)}{E[X(t)^2]} = \frac{R_{xx}(0)}{\sigma_x^2 + \mu_x^2}
\]

Therefore,

\[
Y(t) = X(t) - \frac{R_{xx}(\tau)}{E[X(t)^2]} \cdot X(t + \tau)
\]

Text Note: Equ (6-5) if a zero mean value is assumed for \( X \).

\[
\rho = \frac{R_{xx}(\tau)}{E[X(t)^2]} = \frac{R_{xx}(\tau)}{\sigma_x^2}
\]

Interpretation of the autocorrelation …

The statistical (or probabilistic) similarity of future (or past) samples of a random sample function to itself for an ergodic random process.

How similar is a time shifted version of a time function to itself?
Exercise 6-1.1

A random process has a sample function of the form

\[ X(t) = \begin{cases} 
A, & 0 \leq t \leq 1 \\
0, & \text{else} 
\end{cases} \]

where A is a random variable that is uniformly distributed from 0 to 10.

Find the autocorrelation of the process.

\[ f(a) = \frac{1}{10}, \quad \text{for } 0 \leq a \leq 10 \]

Using

\[ R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] \]

\[ R_{XX}(t_1, t_2) = E[A^2] \quad \text{for } 0 \leq t_1, t_2 \leq 1 \]

\[ R_{XX}(t_1, t_2) = \int_0^{10} \frac{1}{10} \cdot a^2 \cdot da, \quad \text{for } 0 \leq t_1, t_2 \leq 1 \]

\[ R_{XX}(t_1, t_2) = \frac{a^3}{30} \bigg|_0^{10}, \quad \text{for } 0 \leq t_1, t_2 \leq 1 \]

\[ R_{XX}(t_1, t_2) = \frac{1000}{30} = \frac{100}{3}, \quad \text{for } 0 \leq t_1, t_2 \leq 1 \]
Exercise 6-1.2

Define a random variable $Z(t)$ as

$$Z(t) = X(t) + X(t + \tau_1)$$

where $X(t)$ is a sample function from a stationary random process whose autocorrelation function is

$$R_{xx}(\tau) = \exp(-\tau^2)$$

Write an expressions for the autocorrelation function of $Z(t)$.

$$R_{zz}(t_1, t_2) = E[Z(t_1) \cdot Z(t_2)]$$

$$R_{zz}(t_1, t_2) = E\left[\{X(t_1) + X(t_1 + \tau_1)\} \cdot \{X(t_2) + X(t_2 + \tau_1)\}\right]$$

$$R_{zz}(t_1, t_2) = E[\{X(t_1) \cdot X(t_2) + X(t_1) \cdot X(t_2 + \tau_1) + X(t_1 + \tau_1) \cdot X(t_2) + X(t_1 + \tau_1) \cdot X(t_2 + \tau_1)\}]$$

Note: based on the autocorrelation definition, $\tau = t_2 - t_1$

$$R_{zz}(t_1, t_2) = R_{xx}(t_2 - t_1) + R_{xx}(t_2 + \tau_1 - t_1) + R_{xx}(t_2 - t_1 - \tau_1) + R_{xx}(t_2 - t_1)$$

$$R_{zz}(t_1, t_2) = 2 \cdot R_{xx}(\tau) + R_{xx}(\tau + \tau_1) + R_{xx}(\tau - \tau_1)$$

$$R_{zz}(t_1, t_2) = 2 \cdot \exp(-\tau^2) + \exp(-(\tau + \tau_1)^2) + \exp(-(\tau - \tau_1)^2)$$

Note that for this example, the autocorrelation is strictly a function of tau and not $t_1$ or $t_2$. As a result, we expect that the random variable is stationary.

Helpful in defining a stationary random process:

A stationary random variable will have an autocorrelation that is dependent on the time difference, not the absolute times.
Autocorrelation of a Binary Process

Binary communications signals are discrete, non-deterministic signals with plenty of random variables involved in their generations.

Nominally, the binary bit last for a defined period of time, $t_a$, but initially occurs at a random delay (uniformly distributed across one bit interval $0 \leq t_0 < t_a$). The amplitude may be a fixed constant or a random variable.

![Autocorrelation Diagram](image)

- The bit values are independent, identically distributed with probability $p$ that amplitude is $A$ and $q=1-p$ that amplitude is $-A$.

$$pdf(t_0) = \frac{1}{t_a}, \text{ for } 0 \leq t_0 < t_a$$

Determine the autocorrelation of the bipolar binary sequence, assuming $p=0.5$.

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

For samples more than one period apart, $|t_1 - t_2| > t_a$, we must consider

$$E[a_k \cdot a_j] = p \cdot A \cdot p \cdot A + p \cdot A \cdot (1-p) \cdot (-A) + (1-p) \cdot (-A) \cdot p \cdot A + (1-p) \cdot (-A) \cdot (1-p) \cdot (-A)$$

$$E[a_k \cdot a_j] = A^2 \cdot (p^2 - 2 \cdot p \cdot (1-p) + (1-p)^2)$$

$$E[a_k \cdot a_j] = A^2 \cdot (4 \cdot p^2 - 4 \cdot p + 1)$$

For $p=0.5$

$$E[a_k \cdot a_j] = A^2 \cdot (4 \cdot p^2 - 4 \cdot p + 1) = 0$$


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For samples within one period, \(|t_1 - t_2| \leq t_a\),

\[
E[a_k \cdot a_k] = p \cdot A^2 + (1 - p) \cdot (-A)^2 = A^2
\]

\[
E[a_k \cdot a_{k+1}] = A^2 \cdot (4 \cdot p^2 - 4 \cdot p + 1) = 0
\]

there are two regions to consider, the sample bit overlapping and the area of the next bit. But the overlapping area … should be triangular. Therefore

\[
R_{XX}(\tau) = \frac{1}{t_a} \cdot \int_{\tau - t_a/2}^{\tau + t_a/2} E[a_k \cdot a_{k-1}] \, dt
\]

\[
R_{XX}(\tau) = \frac{1}{t_a} \cdot \int_{\tau - t_a/2}^{\tau + t_a/2} E[a_k \cdot a_k] \, dt,
\]

for \(-t_a \leq \tau \leq 0\)

\[
R_{XX}(\tau) = \frac{1}{t_a} \cdot \int_{\tau - t_a/2}^{\tau + t_a/2} E[a_k \cdot a_{k+1}] \, dt,
\]

for \(0 \leq \tau \leq t_a\)

or

\[
R_{XX}(\tau) = A^2 \cdot \frac{1}{t_a} \cdot \int_{\tau - t_a/2}^{\tau + t_a/2} 1 \, dt,
\]

for \(-t_a \leq \tau \leq 0\)

\[
R_{XX}(\tau) = A^2 \cdot \frac{1}{t_a} \cdot \int_{\tau - t_a/2}^{\tau + t_a/2} 1 \, dt,
\]

for \(0 \leq \tau \leq t_a\)

Therefore

\[
R_{XX}(\tau) = \begin{cases} 
A^2 \cdot \frac{t_a + \tau}{t_a}, & \text{for } -t_a \leq \tau \leq 0 \\
A^2 \cdot \frac{t_a - \tau}{t_a}, & \text{for } 0 \leq \tau \leq t_a
\end{cases}
\]

or recognizing the structure

\[
R_{XX}(\tau) = A^2 \left( 1 - \left| \frac{\tau}{t_a} \right| \right), \text{for } -t_a \leq \tau \leq t_a
\]

This is simply a triangular function with maximum of \(A^2\), extending for a full bit period in both time directions.
For unequal bit probability

\[ R_{xx}(\tau) = \begin{cases} 
A^2 \cdot \left( \frac{|t_a - \tau|}{t_a} + \left( 4 \cdot p^2 - 4 \cdot p + 1 \right) \frac{|\tau|}{t_a} \right), & \text{for } -t_a \leq \tau \leq t_a \\
A^2 \cdot \left( 4 \cdot p^2 - 4 \cdot p + 1 \right), & \text{for } t_a \leq |\tau| 
\end{cases} \]

As there are more of one bit or the other, there is always a positive correlation between bits (the curve is a minimum for \( p=0.5 \)), that peaks to \( A^2 \) at \( \tau = 0 \).

Note that if the amplitude is a random variable, the expected value of the bits must be further evaluated. Such as,

\[ E[a_k \cdot a_k] = \sigma^2 + \mu^2 \]

\[ E[a_k \cdot a_{k+1}] = \mu^2 \]

In general, the autocorrelation of communications signal waveforms is important, particularly when we discuss the power spectral density later in the textbook.
Examples of discrete waveforms used for communications, signal processing, controls, etc.

(a) Unipolar RZ & NRZ, (b) Polar RZ & NRZ, (c) Bipolar NRZ, (d) Split-phase Manchester, (e) Polar quaternary NRZ.


In general, a periodic bipolar “pulse” that is shorter in duration than the pulse period will have the autocorrelation function

\[ R_{XX}(\tau) = A^2 \frac{t_w}{t_p} \left(1 - \left|\frac{\tau}{t_w}\right|\right), \quad \text{for } -t_w \leq \tau \leq t_w \]

for a \( t_w \) width pulse existing in a \( t_p \) time period, assuming that positive and negative levels are equally likely.

Sinusoidal Signal

Another classic is a sinusoidal signal with a random phase, uniformly distributed from 0 to $2\pi$.

$$ x(t) = A \cdot \sin(2 \cdot \pi \cdot f \cdot t + \theta) $$

$$ R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] $$

$$ R_{XX}(t_1, t_2) = E[A \cdot \sin(2 \cdot \pi \cdot f \cdot t_1 + \theta) \cdot A \cdot \sin(2 \cdot \pi \cdot f \cdot t_2 + \theta)] $$

$$ R_{XX}(t_1, t_2) = A^2 \cdot E[\sin(2 \cdot \pi \cdot f \cdot t_1 + \theta) \cdot \sin(2 \cdot \pi \cdot f \cdot t_2 + \theta)] $$

$$ R_{XX}(t_1, t_2) = A^2 \cdot E\left[\frac{1}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot (t_2 - t_1)) - \frac{1}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot (t_2 + t_1) + 2 \cdot \theta)\right] $$

$$ R_{XX}(t_1, t_2) = \frac{A^2}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot (t_2 - t_1)) - \frac{A^2}{2} \cdot E\left[\cos(2 \cdot \pi \cdot f \cdot (t_2 + t_1) + 2 \cdot \theta)\right] $$

$$ R_{XX}(t_1, t_2) = \frac{A^2}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot (t_2 + t_1)) $$

$$ R_{XX}(\tau) = \frac{A^2}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot \tau) $$

Of note is that the phase need only be uniformly distributed over 0 to $\pi$!

Also of note, if the amplitude is an independent random variable, then

$$ R_{XX}(\tau) = \frac{E[A^2]}{2} \cdot \cos(2 \cdot \pi \cdot f \cdot \tau) $$
Properties of Autocorrelation Functions

1) \( R_{XX}(0) = E[x^2] = \bar{x}^2 \) or \( R_{XX}(0) = \langle x(t)^2 \rangle \)

the mean squared value of the random process can be obtained by observing the zeroth lag of the autocorrelation function.

2) \( R_{XX}(\tau) = R_{XX}(-\tau) \)

The autocorrelation function is an even function in time. Only positive (or negative) needs to be computed for an ergodic WSS random process.

Note that:
\[
R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]
\]

Let \( t_1 = t_1 \) and \( t_2 = t_1 + \tau \),
then \( E[X(t_1)X(t_1 + \tau)] = R_{xx}(\tau) \)
and for \( R_{xx}(t_1, t_1 + \tau) = E[X(t_1)X(t_1 + \tau)] = E[X(t_1 + \tau)X(t_1)] = R_{xx}(t_1 + \tau, t_1) = R_{xx}(-\tau) \)

Let \( t_1 = t_2 - \tau \) and \( t_2 = t_2 \),
then \( R_{xx}(t_2 - \tau, t_2) = E[X(t_2 - \tau)X(t_2)] = E[X(t_2)X(t_2 - \tau)] = R_{xx}(-\tau) \)

But, both of the above equations are describing exactly the sample correlation in time!
Therefore, \( R_{XX}(\tau) = R_{XX}(-\tau) \)

3) \( |R_{XX}(\tau)| \leq R_{XX}(0) \)

The autocorrelation function is a maximum at 0. For periodic functions, other values may equal the zeroth lag, but never be larger.

\[
0 \leq E\left[ (X_1 \pm X_2)^2 \right] = E\left[ X_1^2 \pm 2 \cdot X_1 \cdot X_2 + X_2^2 \right] \\
0 \leq E\left[ X_1^2 \right] \pm 2 \cdot E\left[ X_1 \cdot X_2 \right] + E\left[ X_2^2 \right] \\
0 \leq R_{XX}(0) \pm 2 \cdot R_{XX}(\tau) + R_{XX}(0)
\]

For the worst case
\[
0 \leq 2 \cdot R_{XX}(0) - 2 \cdot |R_{XX}(\tau)| \\
|R_{XX}(\tau)| \leq R_{XX}(0)
\]
4) If X has a DC component, then \( R_{XX} \) has a constant factor.

\[
X(t) = \bar{X} + N(t)
\]

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[(\bar{X} + N(t)) \cdot (\bar{X} + N(t+\tau))]
\]

\[
R_{XX}(\tau) = E[\bar{X}^2 + \bar{X} \cdot N(t+\tau) + N(t) \cdot \bar{X} + N(t) \cdot N(t+\tau)]
\]

\[
R_{XX}(\tau) = \bar{X}^2 + 2 \cdot \bar{X} \cdot \bar{N} + R_{NN}(\tau)
\]

Note that the mean value can be computed from the autocorrelation function constants!

5) If X has a periodic component, then \( R_{XX} \) has a will also have a periodic component of the same period.

Think of:

\[
X(t) = A \cdot \cos(w \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi
\]

where \( A \) and \( w \) are known constants and theta is a uniform random variable.

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[(A \cdot \cos(w \cdot t + \theta)) \cdot (A \cdot \cos(w \cdot t + w \cdot \tau + \theta))]
\]

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = A^2 \cdot E\left[\frac{1}{2} \cdot \left(\cos(2 \cdot w \cdot t + w \cdot \tau + 2 \cdot \theta) + \cos(w \cdot \tau)\right)\right]
\]

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2} \cdot \{E[\cos(2 \cdot w \cdot t + w \cdot \tau + 2 \cdot \theta)] + E[\cos(w \cdot \tau)]\}
\]

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2} \cdot \left\{\cos(w \cdot \tau) + E\left[\int_{0}^{2\pi} \cos(2 \cdot w \cdot t + w \cdot \tau + 2 \cdot \theta) \cdot \frac{1}{2\pi} \cdot d\theta\right]\right\}
\]

\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2} \cdot \{\cos(w \cdot \tau) + 0\} = \frac{A^2}{2} \cdot \cos(w \cdot \tau)
\]
For a more general case where

\[ X(t) = A \cdot \cos(w \cdot t + \theta) + N(t), \quad 0 \leq \theta \leq 2 \cdot \pi \]

where \( A \) and \( w \) are known constants, \( \theta \) is a uniform random variable and \( N \) is a random variable with \( \theta \) and \( N \) independent.

\[ R_{XX}(\tau) = \frac{A^2}{2} \cdot \cos(\omega \cdot \tau) + R_{NN}(\tau) \]

5b) For signals that are the sum of independent random variable, the autocorrelation is the sum of the individual autocorrelation functions.

\[ W(t) = X(t) + Y(t) \]
\[ R_{WW}(\tau) = E[W(t)W(t + \tau)] = E[(X(t) + Y(t)) \cdot (X(t + \tau) + Y(t + \tau))] \]
\[ R_{WW}(\tau) = E[X(t) \cdot X(t + \tau) + X(t) \cdot Y(t + \tau) + Y(t) \cdot X(t + \tau) + Y(t) \cdot Y(t + \tau)] \]
\[ R_{WW}(\tau) = E[X(t)] \cdot E[Y(t + \tau)] + E[Y(t)] \cdot E[X(t + \tau)] + R_{YY}(\tau) \]
\[ R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2 \cdot \mu_X \cdot \mu_Y \]

For non-zero mean functions, (let \( w, x, y \) be zero mean and \( W, X, Y \) have a mean)

\[ R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2 \cdot \mu_X \cdot \mu_Y \]
\[ R_{WW}(\tau) = R_{ww}(\tau) + \mu_w^2 = R_{xx}(\tau) + \mu_X^2 + R_{yy}(\tau) + \mu_Y^2 + 2 \cdot \mu_X \cdot \mu_Y \]
\[ R_{WW}(\tau) = R_{ww}(\tau) + \mu_w^2 = R_{xx}(\tau) + R_{yy}(\tau) + \mu_X^2 + 2 \cdot \mu_X \cdot \mu_Y + \mu_Y^2 \]
\[ R_{WW}(\tau) = R_{ww}(\tau) + \mu_w^2 = R_{xx}(\tau) + R_{yy}(\tau) + (\mu_X + \mu_Y)^2 \]
\[ R_{ww}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) \]
6) If $X$ is ergodic and zero mean and has no periodic component, then

$$\lim_{|\tau| \to \infty} R_{XX}(\tau) = 0$$

No good proof provided.

The logical argument is that eventually new samples of a non-deterministic random process will become uncorrelated. Once the samples are uncorrelated, the autocorrelation goes to zero.

7) Autocorrelation functions can not have an arbitrary shape. One way of specifying shapes permissible is in terms of the Fourier transform of the autocorrelation function. That is, if

$$\mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-jwt) \cdot dt$$

then the restriction states that

$$\mathcal{F}[R_{XX}(\tau)](w) \geq 0 \quad \text{for all } w$$

Fact affecting this concept: (a) the autocorrelation is an even function, therefore the Fourier transform has a symmetric real part and anti-symmetric imaginary part (b) since it is symmetric, the Fourier coefficients that describe the autocorrelation must be zero for all sin and non-zero for cosine terms.

Additional concept:

$$X(t) = a \cdot N(t)$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[(a \cdot N(t))(a \cdot N(t+\tau))]$$

$$R_{XX}(\tau) = a^2 \cdot E[N(t) \cdot N(t+\tau)] = a^2 \cdot R_{NN}(\tau)$$