Chapter 6: Correlation Functions

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Concepts

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- Linear Transformations
- Properties of Autocorrelation Functions
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- Correlation Matrices For Sampled Functions
Chapter 6: Correlation Functions

The Autocorrelation Function

For a sample function defined by samples in time of a random process, how alike are the different samples?

Define: \( X_1 = X(t_1) \) and \( X_2 = X(t_2) \)

The autocorrelation is defined as:

\[
R_{XX}(t_1, t_2) = E[X_1X_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdot \{x_1x_2, f(x_1, x_2)\}
\]

The above function is valid for all processes, stationary and non-stationary.

For WSS processes:

\[
R_{XX}(t_1, t_2) = E[X(t)X(t+\tau)] = R_{XX}(\tau)
\]

If the process is ergodic, the time average is equivalent to the probabilistic expectation, or

\[
\mathcal{R}_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t+\tau) \cdot dt = \langle x(t) \cdot x(t+\tau) \rangle
\]

and

\[
\mathcal{R}_{XX}(\tau) = R_{XX}(\tau)
\]

The mathematics require correlation as compared to convolution.

Convolution

\[
y(\tau) = \int_{-\infty}^{\infty} h(\lambda) \cdot x(\tau-\lambda) \cdot d\lambda \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)
\]

Correlation

\[
y(\tau) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(\tau+\lambda) \cdot d\lambda \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n+k)
\]
Properties of Autocorrelation Functions

1) \( R_{XX}(0) = E[X^2] = \overline{X^2} \) or \( R_{XX}(0) = \langle x(t)^2 \rangle \)

The mean squared value of the random process can be obtained by observing the zeroth lag of the autocorrelation function.

2) \( R_{XX}(\tau) = R_{XX}(-\tau) \)

The autocorrelation function is an even function in time. Only positive (or negative) needs to be computed for an ergodic, WSS random process.

3) \( |R_{XX}(\tau)| \leq R_{XX}(0) \)

The autocorrelation function is a maximum at 0. For periodic functions, other values may equal the zeroth lag, but never be larger.

4) If \( X \) has a DC component, then \( R_{XX} \) has a constant factor.

\[ X(t) = \overline{X} + N(t) \]

\[ R_{XX}(\tau) = \overline{X^2} + R_{NN}(\tau) \]

Note that the mean value can be computed from the autocorrelation function constants!

5) If \( X \) has a periodic component, then \( R_{XX} \) will also have a periodic component of the same period. For,

\[ X(t) = A \cdot \cos(w \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi \]

where \( A \) and \( w \) are known constants and \( \theta \) is a uniform random variable.

\[ R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2} \cdot \cos(w \cdot \tau) \]

5b) For signals that are the sum of independent random variable, the autocorrelation is the sum of the individual autocorrelation functions.

\[ W(t) = X(t) + Y(t) \]

\[ R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2 \cdot \mu_X \cdot \mu_Y \]

For non-zero mean functions, (let \( w, x, y \) be zero mean and \( W, X, Y \) have a mean)

\[ \mu^2_w = (\mu_x + \mu_y)^2 \]

\[ R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) \]
6)  If $X$ is ergodic and zero mean and has no periodic component, then
\[
\lim_{|\tau| \to \infty} R_{XX}(\tau) = 0
\]
No good proof provided.

The logical argument is that eventually new samples of a non-deterministic random process will become uncorrelated. Once the samples are uncorrelated, the autocorrelation goes to zero.

7)  Autocorrelation functions cannot have an arbitrary shape. One way of specifying shapes permissible is in terms of the Fourier transform of the autocorrelation function. That is, if
\[
\mathbb{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-jwt) \cdot dt
\]
then the restriction states that
\[
\mathbb{F}[R_{XX}(\tau)] \geq 0 \quad \text{for all } w
\]
Additional concepts that are useful … dealing with constant:
\[
X(t) = a \cdot N(t)
\]
\[
R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[(a \cdot N(t)) \cdot (a \cdot N(t+\tau))]
\]
\[
R_{XX}(\tau) = a^2 \cdot E[N(t) \cdot N(t+\tau)] = a^2 \cdot R_{NN}(\tau)
\]
\[
X(t) = a + Y(t)
\]
\[
R_{XX}(\tau) = E[(a + Y(t)) \cdot (a + Y(t+\tau))]
\]
\[
R_{XX}(\tau) = E[a^2 + a \cdot Y(t) + a \cdot Y(t+\tau) + Y(t) \cdot Y(t+\tau)]
\]
\[
R_{XX}(\tau) = a^2 + a \cdot E[Y(t)] + E[Y(t+\tau)] + E[Y(t) \cdot Y(t+\tau)] + R_{YY}(\tau)
\]
\[
R_{XX}(\tau) = a^2 + 2 \cdot a \cdot E[Y(t)] + R_{YY}(\tau)
\]
Autocorrelation of a Binary Process

Binary communications signals are discrete, non-deterministic signals with plenty of random variables involved in their generations.

Nominally, the binary bit last for a defined period of time, $t_a$, but initially occurs at a random delay (uniformly distributed across one bit interval $0 \leq t_0 < t_a$). The amplitude may be a fixed constant or a random variable.

The bit values are independent, identically distributed with probability $p$ that amplitude is $A$ and $q=1-p$ that amplitude is $-A$.

\[ \text{pdf}(t_0) = \frac{1}{t_a}, \quad \text{for } 0 \leq t_0 < t_a \]

Determine the autocorrelation of the bipolar binary sequence, assuming $p=0.5$.

For $p=0.5$

\[ E[a_k \cdot a_j] = A^2 \cdot (4 \cdot p^2 - 4 \cdot p + 1) = 0 \]

\[ R_{XX}(\tau) = A^2 \cdot \left( 1 - \frac{|\tau|}{t_a} \right), \quad \text{for } -t_a \leq \tau \leq t_a \]

This is simply a triangular function with maximum of $A^2$, extending for a full bit period in both time directions.
For unequal bit probability

\[
R_{x(t)}(\tau) = \begin{cases} 
A^2 \cdot \left( \frac{t_a - |r|}{t_a} + (4 \cdot p^2 - 4 \cdot p + 1) \frac{|r|}{t_a} \right), & \text{for } -t_a \leq \tau \leq t_a \\
A^2 \cdot (4 \cdot p^2 - 4 \cdot p + 1), & \text{for } t_a \leq |r|
\end{cases}
\]

As there are more of one bit or the other, there is always a positive correlation between bits (the curve is a minimum for p=0.5), that peaks to \(A^2\) at \(\tau = 0\).

Note that if the amplitude is a random variable, the expected value of the bits must be further evaluated. Such as,

\[
E[a_k \cdot a_k] = \sigma^2 + \mu^2
\]

\[
E[a_k \cdot a_{k+1}] = \mu^2
\]

In general, the autocorrelation of communications signal waveforms is important, particularly when we discuss the power spectral density later in the textbook.

Examples of discrete waveforms used for communications, signal processing, controls, etc.

(a) Unipolar RZ & NRZ, (b) Polar RZ & NRZ, (c) Bipolar NRZ, (d) Split-phase Manchester, (e) Polar quaternary NRZ.

Exercise 6-3.1

a) An ergodic random process has an autocorrelation function of the form

\[ R_{xx}(\tau) = 9 \cdot \exp(-4 \cdot |\tau|) + 16 \cdot \cos(10 \cdot \tau) + 16 \]

Find the mean-square value, the mean value, and the variance of the process.

The mean-square (2nd moment) is

\[ E[X^2] = R_{xx}(0) = 9 + 16 + 16 = 41 = \sigma^2 + \mu^2 \]

The constant portion of the autocorrelation represents the square of the mean. Therefore

\[ E[X] = \mu = 16 \quad \text{and} \quad \mu = \pm 4 \]

Finally, the variance can be computed as,

\[ \sigma^2 = E[X^2] - E[X]^2 = R_{xx}(0) - \mu^2 = 41 - 16 = 25 \]

b) An ergodic random process has an autocorrelation function of the form

\[ R_{xx}(\tau) = \frac{4 \cdot \tau^2 + 6}{\tau^2 + 1} \]

Find the mean-square value, the mean value, and the variance of the process.

The mean-square (2nd moment) is

\[ E[X^2] = R_{xx}(0) = \frac{6}{1} = 6 = \sigma^2 + \mu^2 \]

The constant portion of the autocorrelation represents the square of the mean. Therefore

\[ E[X]^2 = \mu = \lim_{\tau \to \infty} \frac{4 \cdot \tau^2 + 6}{\tau^2 + 1} = \frac{4}{1} = 4 \quad \text{and} \quad \mu = \pm 2 \]

Finally, the variance can be computed as,

\[ \sigma^2 = E[X^2] - E[X]^2 = R_{xx}(0) - \mu^2 = 6 - 4 = 2 \]
Another Autocorrelation examples, using a time average

Example: Rect. Function

From Dr. Severance’s notes;

\[ X(t) = \sum_{k} A_k \cdot \text{Re} \left( \frac{t-t_0 + kT}{T} \right) \]

where \(A(k)\) are zero mean, identically distributed, independent R.V., \(t_0\) is a R.V. uniformly distributed over possible bit period \(T\), and \(T\) is a constant. \(k\) describes the bit positions.

\[
R_{XX}(t,t+\tau) = E \left[ \sum_{k} A_k \cdot \text{Re} \left( \frac{t-t_0 + kT}{T} \right) \right] \cdot \sum_{j} A_j \cdot \text{Re} \left( \frac{t + \tau - t_0 + jT}{T} \right)
\]

\[
R_{XX}(t,t+\tau) = \sum_{k} \cdot \sum_{j} E \left[ A_k \cdot A_j \cdot \text{Re} \left( \frac{t-t_0 + kT}{T} \right) \cdot \text{Re} \left( \frac{t + \tau - t_0 + jT}{T} \right) \right]
\]

but

\[
E[A_k \cdot A_j] = \begin{cases} E[A_k^2] = \sigma_A^2 + \mu_A^2 & \text{for } k = j \\ E[A_k^2] = \mu_A^2 & \text{for } k \neq j \end{cases}
\]

If we assume that the symbol has a zero mean \(\mu_A = 0 \ldots\) and

Therefore (one summation goes away)

\[
R_{XX}(t,t+\tau) = E[A_k^2] \cdot \sum_{k} E \left[ \text{Re} \left( \frac{t-t_0 + kT}{T} \right) \cdot \text{Re} \left( \frac{t + \tau - t_0 + kT}{T} \right) \right]
\]

Now using the time average concept

\[
\Re_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot x(t+\tau) \cdot dt = \langle x(t) \cdot x(t+\tau) \rangle
\]

Establishing the time average as the average of the kth bit, (notice that the delay is incorporated into the integral, and assuming that the signal is Wide-Sense Stationary)
\[
\Re XX (\tau) = E[A^2] \cdot \frac{1}{T} \cdot \int_{-T/2 + t_0}^{T/2 + t_0} \left[ 1 \cdot \text{Rect} \left( \frac{t + \tau}{T} \right) \right] \, dt
\]

for \( T > \tau = 0 \)

\[
\Re XX (\tau) = E[A^2] \cdot \frac{1}{T} \cdot \int_{-T/2 + t_0 + \tau}^{T/2 + t_0 + \tau} [1 \cdot 1] \, dt = E[A^2] \cdot \frac{1}{T} \cdot \int_{T/2 + t_0}^{T/2 + t_0 + \tau} [1] \, dt
\]

\[
\Re XX (\tau) = E[A^2] \cdot \frac{1}{T} \cdot \left[ (T/2 + t_0) - (T/2 + t_0 + \tau) \right] = E[A^2] \cdot \frac{1}{T} \cdot [T - \tau]
\]

\[
\Re XX (\tau) = E[A^2] \cdot \left[ 1 - \frac{\tau}{T} \right], \quad \text{for } \tau \geq 0
\]

Due to symmetry, \( \Re XX (\tau) = \Re XX (-\tau) \), we have

\[
\Re XX (\tau) = E[A^2] \cdot \left[ 1 - \frac{\lvert \tau \rvert}{T} \right]
\]

Note: if the symbol has a mean, there are additional “triangles” at all the offset bit times in \( T \).
When the “triangles” are all summed a DC level based on the mean of the symbol squared will result. Therefore,

\[
E[A_k \cdot A_j] = \begin{cases} E[A^2] = \sigma^2_a + \mu^2_a & \text{for } k = j \\ E[A] = \mu_a & \text{for } k \neq j \end{cases}
\]

And

\[
\Re XX (\tau) = \left[ E[A^2] + E[A]^2 \right] \cdot \left[ 1 - \frac{\lvert \tau \rvert}{T} \right] + E[A]^2 \cdot \left[ 1 - \frac{\lvert \tau \pm n \cdot T \rvert}{T} \right] \quad \text{for } n \neq 0
\]

\[
\Re XX (\tau) = E[A^2] \cdot \left[ 1 - \frac{\lvert \tau \rvert}{T} \right] + E[A]^2
\]
Example: Rect. Function that does not extend for the full period

\[ X(t) = \sum_{k} A_k \cdot \text{Re} \left( \frac{t - t_0 + kT}{\alpha \cdot T} \right) \]

where \( A(k) \) are zero mean, identically distributed, independent R.V., \( t_0 \) is a R.V. uniformly distributed over possible bit period \( T \), and \( T \) and alpha are constants. \( k \) describes the bit positions.

From the previous example, the average value during the time period was defined as:

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \frac{1}{T} \cdot \int_{-T/2 + t_0}^{T/2 + t_0} \left[ 1 \cdot \text{Re} \left( \frac{t + \tau}{T} \right) \right] \cdot dt \]

For this example, the period remains the same, but the integration period due to the rect functions change to:

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \frac{1}{T} \cdot \int_{-\alpha T/2 + t_0 + \tau}^{\alpha T/2 + t_0} \left[ 1 \cdot \text{Re} \left( \frac{t + \tau}{\alpha T} \right) \right] \cdot dt \]

for \( T > \tau = 0 \)

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \frac{1}{T} \cdot \int_{-\alpha T/2 + t_0 + \tau}^{\alpha T/2 + t_0} \left[ 1 \right] \cdot dt = E\left[A^2\right] \cdot \frac{1}{T} \cdot \left[ t \right]_{-\alpha T/2 + t_0 + \tau}^{\alpha T/2 + t_0 + \tau} \]

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \frac{1}{T} \cdot \left[ \alpha T/2 + t_0 - \alpha T/2 + t_0 + \tau \right] = E\left[A^2\right] \cdot \frac{1}{T} \cdot \left[ \alpha T - \tau \right] \]

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \frac{\alpha T}{T} \cdot \left[ 1 - \frac{\tau}{\alpha T} \right] = E\left[A^2\right] \cdot \alpha \cdot \left[ 1 - \frac{\tau}{\alpha T} \right], \quad \text{for } \tau \geq 0 \]

Due to symmetry, we have

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \alpha \cdot \left[ 1 - \frac{\vert \tau \vert}{\alpha T} \right] \]

Note: if the symbol has a mean, there are additional “triangles” at all the offset bit times in \( T \).

\[ \mathbb{R}_{XX}(\tau) = E\left[A^2\right] \cdot \alpha \cdot \left[ 1 - \frac{\vert \tau \vert}{\alpha T} \right] + E\left[A^2\right] \cdot \alpha \cdot \left[ 1 - \frac{\vert \tau \pm n \cdot T \vert}{\alpha T} \right] \]
The Crosscorrelation Function

For a two sample function defined by samples in time of two random processes, how alike are the different samples?

Define: \( X_1 = X(t_1) \) and \( Y_2 = Y(t_2) \)

The crosscorrelation is defined as:

\[
R_{XY}(t_1, t_2) = E[X_1 Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_2 \cdot \{x_1 y_2 f(x_1, y_2)\}
\]

\[
R_{YX}(t_1, t_2) = E[Y_1 X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \cdot \{y_1 x_2 f(y_1, x_2)\}
\]

The above function is valid for all processes, jointly stationary and non-stationary.

For jointly WSS processes:

\[
R_{XY}(t_1, t_2) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)
\]

\[
R_{YX}(t_1, t_2) = E[Y(t)X(t + \tau)] = R_{YX}(\tau)
\]

Note: the order of the subscripts is important for cross-correlation!

If the processes are jointly ergodic, the time average is equivalent to the probabilistic expectation, or

\[
\mathbb{R}_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot y(t + \tau) \cdot dt = \langle x(t) \cdot y(t + \tau) \rangle
\]

\[
\mathbb{R}_{YX}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t) \cdot x(t + \tau) \cdot dt = \langle y(t) \cdot x(t + \tau) \rangle
\]

and

\[
\mathbb{R}_{XY}(\tau) = R_{XY}(\tau)
\]

\[
\mathbb{R}_{YX}(\tau) = R_{YX}(\tau)
\]
Properties of Crosscorrelation Functions

1) The properties of the zeroth lag have no particular significance and do not represent mean-square values. It is true that the “ordered” crosscorrelations must be equal at 0.

\[ R_{XY}(0) = R_{YX}(0) \quad \text{or} \quad R_{XY}(0) = R_{YX}(0) \]

2) Crosscorrelation functions are not generally even functions. However, there is an antisymmetry to the ordered crosscorrelations:

\[ R_{XY}(\tau) = R_{YX}(-\tau) \]

For

\[ R_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot y(t + \tau) \cdot dt = \langle x(t) \cdot y(t + \tau) \rangle \]

Substitute \( t + \tau = \eta \)

\[ R_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+\tau}^{T+\tau} x(\eta - \tau) \cdot y(\eta) \cdot d\eta = \langle x(\eta - \tau) \cdot y(\eta) \rangle \]

\[ R_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+\tau}^{T+\tau} y(\eta) \cdot x(\eta - \tau) \cdot d\eta = \langle y(\eta) \cdot x(\eta - \tau) \rangle = R_{YX}(-\tau) \]

3) The crosscorrelation does not necessarily have its maximum at the zeroth lag. This makes sense if you are correlating a signal with a timed delayed version of itself. The crosscorrelation should be a maximum when the lag equals the time delay!

It can be shown however that \( |R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) \cdot R_{YY}(0)} \)

As a note, the crosscorrelation may not achieve the maximum anywhere …

4) If \( X \) and \( Y \) are statistically independent, then the ordering is not important

\[ R_{XY}(\tau) = E[X(t) \cdot Y(t + \tau)] = E[X(t)] \cdot E[Y(t + \tau)] = \overline{X} \cdot \overline{Y} \]

and

\[ R_{XY}(\tau) = \overline{X} \cdot \overline{Y} = R_{YX}(\tau) \]
5) If $X$ is a stationary random process and is differentiable with respect to time, the crosscorrelation of the signal and its derivative is given by

$$R_{X\dot{X}}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

Defining derivation as a limit:

$$\dot{X}(\tau) = \lim_{e \to 0} \frac{X(t + e) - X(t)}{e}$$

and the crosscorrelation

$$R_{X\dot{X}}(\tau) = E[X(t) \cdot \dot{X}(t + \tau)] = E\left[X(t) \cdot \left(\lim_{e \to 0} \frac{X(t + \tau + e) - X(t + \tau)}{e}\right)\right]$$

$$R_{X\dot{X}}(\tau) = \lim_{e \to 0} \frac{E[X(t) \cdot X(t + \tau + e) - X(t) \cdot X(t + \tau)]}{e}$$

$$R_{X\dot{X}}(\tau) = \lim_{e \to 0} \frac{E[X(t) \cdot X(t + \tau)] - E[X(t) \cdot X(t + \tau)]}{e}$$

$$R_{X\dot{X}}(\tau) = \lim_{e \to 0} \frac{R_{X\dot{X}}(\tau + e) - R_{X\dot{X}}(\tau)}{e}$$

$$R_{X\dot{X}}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

Similarly,

$$R_{\dot{X}\ddot{X}}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

That is all the properties for the crosscorrelation function.
Example: A delayed signal in noise … cross correlated with the original signal …

\[ X(t) \]
\[ Y(t) = a \cdot X(t - t_d) + N(t) \]

Find

\[ R_{XY}(\tau) = E[X(t)Y(t + \tau)] \]

Then

\[ R_{XY}(\tau) = E[X(t) \cdot (a \cdot X(t - t_d + \tau) + N(t + \tau))] \]
\[ R_{XY}(\tau) = a \cdot R_{XX}(\tau - t_d) + R_{YN}(\tau) \]

For noise independent of the signal,

\[ R_{YN}(\tau) = \overline{X} \cdot \overline{N} = R_{NX}(\tau) \]

and therefore

\[ R_{XY}(\tau) = a \cdot R_{XX}(\tau - t_d) + \overline{X} \cdot \overline{N} \]

For either X or N a zero mean process, we would have

\[ R_{XY}(\tau) = a \cdot R_{XX}(\tau - t_d) \]

Does this suggest a way to determine the time delay based on what you know of the autocorrelation?

The maximum of the autocorrelation is at zero; therefore, the maximum of the crosscorrelation can reveal the time delay!

- any electronic system based on the time delay of a known, transmitted signal pulse …
- Applications: radar, sonar, digital communications, etc.
- Note: in communications, one form of “optimal” detector for a digital communications symbol is a correlation receiver/detector. One detector “perfectly matched” to each symbol transmitted is defined. The detector with the maximum output at the correct time delay is selected as the symbol that was sent!
Measurement of the Autocorrelation Function

We love to use time average for everything. For wide-sense stationary, ergodic random processes, time average are equivalent to statistical or probability based values.

$$\mathbb{R}_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t + \tau) \cdot dt = \langle x(t) \cdot x(t + \tau) \rangle$$

Using this fact, how can we use short-term time averages to generate auto- or cross-correlation functions?

An estimate of the autocorrelation is defined as:

$$\hat{R}_{XX}(\tau) = \frac{1}{T - \tau} \int_{0}^{T - \tau} x(t) \cdot x(t + \tau) \cdot dt, \quad \text{for} \quad 0 \leq \tau \ll T$$

Note that the time average is performed across as much of the signal that is available after the time shift by tau.

In most practical cases, the function is performed in terms of digital samples taken at specific time intervals, $\Delta t$. For tau based on the available time step, $k$, with $N$ equating to the available time interval, we have:

$$\hat{R}_{XX}(k\Delta t) = \frac{1}{((N + 1)\Delta t) - (k\Delta t)} \sum_{i=0}^{N-k} x(i\Delta t) \cdot x(i\Delta t + k\Delta t) \cdot \Delta t$$

$$\hat{R}_{XX}(k\Delta t) = \hat{R}_{XX}(k) = \frac{1}{N + 1 - k} \sum_{i=0}^{N-k} x(i) \cdot x(i + k)$$

In computing this autocorrelation, the initial weighting term approaches 1 when $k=N$. At this point the entire summation consists of one point and is therefore a poor estimate of the autocorrelation. For useful results, $k<<N$!
It can be shown (below) that the estimated autocorrelation is equivalent to the actual autocorrelation; therefore, this is an unbiased estimate.

\[
E[\hat{R}_{XX}(k\Delta t)] = E[\hat{R}_{XX}(k)] = E \left[ \frac{1}{N+1-k} \sum_{i=0}^{N-k} x(i) \cdot x(i+k) \right]
\]

\[
E[\hat{R}_{XX}(k\Delta t)] = \frac{1}{N+1-k} \cdot \sum_{i=0}^{N-k} E[x(i) \cdot x(i+k)]
\]

\[
= \frac{1}{N+1-k} \cdot \sum_{i=0}^{N-k} R_{XX}(k\Delta t)
\]

\[
= \frac{1}{N+1-k} \cdot (N-k+1) \cdot R_{XX}(k\Delta t)
\]

\[
E[\hat{R}_{XX}(k\Delta t)] = R_{XX}(k\Delta t)
\]

As noted, the validity of each of the summed autocorrelation lags can and should be brought into question as \( k \) approaches \( N \). As a result, a biased estimate of the autocorrelation is commonly used. The biased estimate is defined as:

\[
\tilde{R}_{XX}(k) = \frac{1}{N+1} \sum_{i=0}^{N-k} x(i) \cdot x(i+k)
\]

Here, a constant weight instead of one based on the number of elements summed is used. This estimate has the property that the estimated autocorrelation should decrease as \( k \) approaches \( N \).

The expected value of this estimate can be shown to be

\[
E[\tilde{R}_{XX}(k)] = \left( 1 - \frac{k}{N+1} \right) \cdot R_{XX}(k\Delta t)
\]

The variance of this estimate can be shown to be (math not done)

\[
Var[\tilde{R}_{XX}(k)] \leq \frac{2}{N} \cdot \sum_{k=-M}^{M} [R_{XX}(k\Delta t)]^2
\]

This equation can be used to estimate the number of time samples needed for a useful estimate.
Matlab Example

```matlab
%%
% Figure 6_2
%%
clear;
close all;

nsamples = 1000;
n=(0:1:nsamples-1)';

%%
% rect
rect_length = 100;

x = zeros(nsamples,1);
x(100+(1:rect_length))=1;

% Scale to make the maximum unity
Rxx1 = xcorr(x)/rect_length;
Rxx2 = xcorr(x,'unbiased');
nn = -(nsamples-1):1:(nsamples-1);

figure
plot(n,x)
figure
plot(nn,Rxx1,nn,Rxx2)
legend('Raw','Unbiased')
```

**Example: Excel**

Skill 26.1

Consider the random process generated by the difference equation

\[
x(k) = 0.90 \cdot x(k-1) - 0.81 \cdot x(k-2) + \sqrt{3} \cdot (2 \cdot RND - 1)
\]

\[
x(0) = 1
\]

\[
x(1) = 1
\]

where RND is a uniformly distributed random number between 0 and 1. This is an autoregressive process in that future values are dependent upon the past values of the signal. Note the values that follow if there was no random input … (1, 1, 0.09, -0.729, -0.729, 0.06561, …)

Assume that the signal is sampled 5 times a second (0.2 sec. interval). Create a spread sheet to generate the signal and compute the autocorrelation of the result.

Note: The frequency response of the “filter” is (based on ECE4550 information).
Example: MATLAB Figure 6-3 and 6-4 – What you see when doing a simulation!

```matlab
%%
% Figure 6_3
% clear;
close all;

rng(1000); % Replaces rand('seed',1000)
x = 10*randn(1, 1001);
t1 = 0:0.001:1;
tlag = (-1:0.001:1);

Rxx = xcorr(x) / (1002);

figure(1)
subplot(2,1,1)
plot(t1, x);
xlabel('time'); ylabel('X');

subplot(2,1,2)
plot(tlag, Rxx);
xlabel('Lag time'); ylabel('Rxx');
```


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%%
% Figure 6_4
% clear;
close all;

rng(1000); % Replaces rand('seed',1000)
x1=10*randn(1,1001);

h=ones(1,51)/51;
x=conv(h,x1);
x2=x(25:25+1000);

t1=0:0.001:1;
tlag=(-1:0.001:1);

Rx1=xcorr(x1)/(1002);
Rx2=xcorr(x2)/(1002);

figure(1)
subplot(2,1,1)
plot(t1,x1);
xlabel('time'); ylabel('X');

subplot(2,1,2)
plot(tlag,Rx1);
xlabel('Lag time'); ylabel('Rxx');

figure(2)
subplot(2,1,1)
plot(t1,x2);
xlabel('time'); ylabel('X');

subplot(2,1,2)
plot(tlag,Rx2);
xlabel('Lag time'); ylabel('Rxx');

%%
% Variance
% M=length(Rx1);
V1=(2/M)*sum(Rx1.^2);
V2=(2/M)*sum(Rx2.^2);
S1=sqrt(V1);
S2=sqrt(V2);

fprintf('Data Variance %g
',V1);
fprintf('Data Standard Deviation %g
',S1);
fprintf('Filter Variance %g
',V2);
fprintf('Filter Standard Deviation %g
',S2);
Data  Variance 22.6494
Data  Standard Deviation 4.75913
Filter Variance 0.110392
Filter Standard Deviation 0.332252
Example Cross-correlation (Fig_6_9)

A signal is measured in the presence of Gaussian noise having a bandwidth of 50 Hz and a standard deviation of $\sqrt{10}$.

$$y(t) = x(t) + n(t)$$

For

$$x(t) = \sqrt{2} \cdot \sin(2 \cdot \pi \cdot 500 \cdot t + \theta)$$

where $\theta$ is uniformly distributed from $[0, 2\pi]$.

The corresponding signal-to-noise ratio, SNR, (in terms of power) can be computed as

$$SNR = \frac{S_{\text{Power}}}{N_{\text{Power}}} = \frac{\frac{A^2}{2}}{\frac{\sigma_n^2}{2}} = \frac{\sqrt{2}^2}{\sqrt{10}^2} = \frac{1}{10} = 0.1$$

$$SNR_{dB} = 10 \cdot \log_{10}(0.1) \approx -10 dB$$

(a) Form the autocorrelation

(b) Form the crosscorrelation with a pure sinusoid, $LO(t) = \sqrt{2} \cdot \sin(2 \cdot \pi \cdot 500 \cdot t)$
Correlation Matrices for Sampled Functions

Collect a vector of time samples:

\[
X = \begin{bmatrix}
X(t_1) \\
X(t_2) \\
\vdots \\
X(t_N)
\end{bmatrix}
\]

Forming the autocorrelation matrix:

\[
R_{XX} = E[X \cdot X^T] = \begin{bmatrix}
X(t_1) \\
X(t_2) \\
\vdots \\
X(t_N)
\end{bmatrix} \cdot \begin{bmatrix}
X(t_1) & X(t_2) & \cdots & X(t_N)
\end{bmatrix}
\]

\[
R_{XX} = \begin{bmatrix}
X(t_1) \cdot X(t_1) & X(t_1) \cdot X(t_2) & \cdots & X(t_1) \cdot X(t_N) \\
X(t_2) \cdot X(t_1) & X(t_2) \cdot X(t_2) & \cdots & X(t_2) \cdot X(t_N) \\
\vdots & \vdots & \ddots & \vdots \\
X(t_N) \cdot X(t_1) & X(t_N) \cdot X(t_2) & \cdots & X(t_N) \cdot X(t_N)
\end{bmatrix}
\]

If the random process is Wide Sense Stationary, the autocorrelation is related to the lag times. Assuming that the data consists of consecutive time samples at a constant interval,

\[
t_2 = t_1 + \Delta t \\
t_k = t_1 + (k-1) \cdot \Delta t
\]

Then

\[
R_{XX} = \begin{bmatrix}
R_{XX}(0) & R_{XX}(\Delta t) & \cdots & R_{XX}((N-1) \cdot \Delta t) \\
R_{XX}(-\Delta t) & R_{XX}(0) & \cdots & R_{XX}((N-2) \cdot \Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
R_{XX}(-(N-1) \cdot \Delta t) & R_{XX}(-(N-2) \cdot \Delta t) & \cdots & R_{XX}(0)
\end{bmatrix}
\]

but since the autocorrelation function is symmetric

\[
R_{XX} = \begin{bmatrix}
R_{XX}(0) & R_{XX}(\Delta t) & \cdots & R_{XX}((N-1) \cdot \Delta t) \\
R_{XX}(\Delta t) & R_{XX}(0) & \cdots & R_{XX}((N-2) \cdot \Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
R_{XX}((N-1) \cdot \Delta t) & R_{XX}((N-2) \cdot \Delta t) & \cdots & R_{XX}(0)
\end{bmatrix}
\]
Note that this matrix takes on a banded form along the diagonal, a Toeplitz Matrix. It has “useful” properties when performing matrix computations!

The covariance matrix can also be described:

\[
C_{XX} = \mathbb{E}[(X - \bar{X}) \cdot (X - \bar{X})^T] = \begin{bmatrix}
X(t_1) - \bar{X} & X(t_2) - \bar{X} & \cdots & X(t_N) - \bar{X}
\end{bmatrix} \begin{bmatrix}
X(t_1) - \bar{X} & \rho_{12} \cdot \sigma_1 \cdot \sigma_2 & \cdots & \rho_{1N} \cdot \sigma_1 \cdot \sigma_N \\
\rho_{21} \cdot \sigma_2 \cdot \sigma_1 & \sigma_2 & \cdots & \sigma_N \cdot \sigma_2 \cdot \sigma_N \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{N1} \cdot \sigma_N \cdot \sigma_1 & \rho_{N2} \cdot \sigma_N \cdot \sigma_2 & \cdots & \sigma_N \cdot \sigma_N
\end{bmatrix}
\]

For a WSS process, all the variances are identical and the correlation coefficients are based on the lag interval.

\[
C_{XX} = \Lambda_{XX} = \sigma^2 \cdot \begin{bmatrix}
1 & \rho_1 & \cdots & \rho_{N-1} \\
\rho_1 & 1 & \cdots & \rho_{N-2} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{N-1} & \rho_{N-2} & \cdots & 1
\end{bmatrix}
\]

**An Example**

Let \( R_{XX}(\tau) = 10 \cdot \exp(-|\tau|) + 9 \)

Generating a 3 x 3 matrix based on time samples that are “1 unit” apart yields:

\[
R_{XX} = \begin{bmatrix}
R_{XX}(0) & R_{XX}(\Delta t) & R_{XX}(2\Delta t) \\
R_{XX}(\Delta t) & R_{XX}(0) & R_{XX}(\Delta t) \\
R_{XX}(2\Delta t) & R_{XX}(\Delta t) & R_{XX}(0)
\end{bmatrix} = \begin{bmatrix}
19 & 12.68 & 10.35 \\
12.68 & 19 & 12.68 \\
10.35 & 12.68 & 19
\end{bmatrix}
\]

\[
C_{XX} = \Lambda_{XX} = R_{XX} - (\bar{X})^2 = \begin{bmatrix}
19 & 12.68 & 10.35 \\
12.68 & 19 & 12.68 \\
10.35 & 12.68 & 19
\end{bmatrix} - 9
\]

\[
C_{XX} = \Lambda_{XX} = 10 \cdot \begin{bmatrix}
10 & 3.68 & 1.35 \\
3.68 & 10 & 3.68 \\
1.35 & 3.68 & 10
\end{bmatrix} = 10 \cdot \begin{bmatrix}
1 & 0.368 & 0.135 \\
0.368 & 1 & 0.368 \\
0.135 & 0.368 & 1
\end{bmatrix} = 10 \cdot \begin{bmatrix}
e^{-1} & e^{-2} & \\
e^{-2} & e^{-1} & 1
\end{bmatrix}
\]


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Welcome to concepts that you can discover more about in graduate school …

For an alternate interpretation, if you could measure multiple random processes, computation of the Autocorrelation matrix would provide the second moments for each of the processes and the correlation of each process to every other process!

\[
X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_N(t) \end{bmatrix}
\]

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \begin{array}{cccc} X_1(t) \cdot X_1(t) & X_1(t) \cdot X_2(t) & \cdots & X_1(t) \cdot X_N(t) \\ X_2(t) \cdot X_1(t) & X_2(t) \cdot X_2(t) & \cdots & X_2(t) \cdot X_N(t) \\ \vdots & \vdots & \ddots & \vdots \\ X_N(t) \cdot X_1(t) & X_N(t) \cdot X_2(t) & \cdots & X_N(t) \cdot X_N(t) \end{array} \right] \cdot dt
\]

\[
R_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \begin{array}{cccc} X_1(t) \cdot X_1(t) & X_1(t) \cdot X_2(t) & \cdots & X_1(t) \cdot X_N(t) \\ X_2(t) \cdot X_1(t) & X_2(t) \cdot X_2(t) & \cdots & X_2(t) \cdot X_N(t) \\ \vdots & \vdots & \ddots & \vdots \\ X_N(t) \cdot X_1(t) & X_N(t) \cdot X_2(t) & \cdots & X_N(t) \cdot X_N(t) \end{array} \right] \cdot dt
\]

\[
R_{XX} = \frac{1}{2N+1} \sum_{i=-N}^{N} \left[ \begin{array}{cccc} X_1(i) \cdot X_1(i) & X_1(i) \cdot X_2(i) & \cdots & X_1(i) \cdot X_N(i) \\ X_2(i) \cdot X_1(i) & X_2(i) \cdot X_2(i) & \cdots & X_2(i) \cdot X_N(i) \\ \vdots & \vdots & \ddots & \vdots \\ X_N(i) \cdot X_1(i) & X_N(i) \cdot X_2(i) & \cdots & X_N(i) \cdot X_N(i) \end{array} \right]
\]

This concept is particularly useful when multiple sensors are measuring the same physical phenomenon. Using multiple sensors, the noise in each sensor is independent, but the signal-of-interest is not. The correlations formed can be used to describe the relative strength of the SOI between sensors, while the autocorrelations relate the signal plus noise power observed in each sensor.
This concept may also be applied for cross correlation, except the matrix dimensions may no longer be square.

Let \( X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_N(t) \end{bmatrix} \) and \( Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_M(t) \end{bmatrix} \)

The resulting cross-correlation matrix is now \( N \times M \).

\[
R_{XY} = E[X \cdot Y^T] = \begin{bmatrix}
X_1(t) \cdot Y_1(t) & X_1(t) \cdot Y_2(t) & \cdots & X_1(t) \cdot Y_M(t) \\
X_2(t) \cdot Y_1(t) & X_2(t) \cdot Y_2(t) & \cdots & X_2(t) \cdot Y_M(t) \\
\vdots & \vdots & \ddots & \vdots \\
X_N(t) \cdot Y_1(t) & X_N(t) \cdot Y_2(t) & \cdots & X_N(t) \cdot Y_M(t)
\end{bmatrix}
\]

As an example, imagine that there are \( N \) systems that consist of noise and a signal at a unique frequency. If the signals are cross correlated with a reference signal with a desired frequency, you would expect only the element containing that frequency to be non-zero.

Let \( X(t) = \begin{bmatrix} A \cdot \exp(\frac{j2\pi f_1}{f_s}t + j\theta_1) + N_1 \\ A \cdot \exp(\frac{j2\pi f_2}{f_s}t + j\theta_2) + N_2 \\ \vdots \\ A \cdot \exp(\frac{j2\pi f_3}{f_s}t + j\theta_3) + N_3 \end{bmatrix} \) and \( Y(t) = \exp(\frac{j2\pi f_d}{f_s}t) \)

\[
R_{XY} = \frac{1}{2N+1} \sum_{i=-N}^{N} \begin{bmatrix}
X_1(i) \cdot \text{conj}(Y(i)) \\
X_2(i) \cdot \text{conj}(Y(i)) \\
\vdots \\
X_N(i) \cdot \text{conj}(Y(i))
\end{bmatrix}
\]

\[
R_{XY} = \frac{1}{2N+1} \sum_{i=-N}^{N} \begin{bmatrix}
A \cdot \exp\left(\frac{j2\pi(f_1 - f_d)}{f_s}t + j\theta_1\right) + N_1' \\
A \cdot \exp\left(\frac{j2\pi(f_2 - f_d)}{f_s}t + j\theta_2\right) + N_2' \\
\vdots \\
A \cdot \exp\left(\frac{j2\pi(f_3 - f_d)}{f_s}t + j\theta_3\right) + N_3'
\end{bmatrix}
\]

Where the frequency “cancels” a value will remain, where they don’t it should sum to zero in time (assuming the noise is zero mean).
As another alternative, if the same signal is delayed getting to each of multiple antenna the this would turn into (notice that the phase and frequencies for $X$ are now the same for all signals)

$$R_{XY} = \frac{1}{2N+1} \sum_{i=-N}^{N} \left[ A \cdot \exp \left( j2\pi \left( f_1 - f_d \right) \frac{i}{f_s} + j2\pi f_1 \tau_1 + j\theta_1 \right) + N_1 \right]$$

$$+ \left[ A \cdot \exp \left( j2\pi \left( f_1 - f_d \right) \frac{i}{f_s} + j2\pi f_1 \tau_2 + j\theta_1 \right) + N_2 \right]$$

$$+ \cdots$$

$$+ \left[ A \cdot \exp \left( j2\pi \left( f_1 - f_d \right) \frac{i}{f_s} + j2\pi f_1 \tau_3 + j\theta_1 \right) + N_3 \right]$$

If you “tune correctly, you get

$$R_{XY} = \frac{1}{2N+1} \sum_{i=-N}^{N} \left[ A \cdot \exp \left( j2\pi f_1 \tau_1 + j\theta_1 \right) + N_1 \right]$$

$$+ \left[ A \cdot \exp \left( j2\pi f_1 \tau_2 + j\theta_1 \right) + N_2 \right]$$

$$+ \cdots$$

$$+ \left[ A \cdot \exp \left( j2\pi f_1 \tau_3 + j\theta_1 \right) + N_3 \right]$$

and the relative phases between each of the antenna elements is related to the differential arrival time between the antenna and reference. What would happen if you could change the phase in each of the antenna inputs to perfectly align … and then sum them together.

This is the concept of Beamforming as applied in “Smart Antenna” systems.

More Examples

Matlab

Generation and correlation of a BPSK signal ….

Is it really a triangle ….