Chapter 7: Spectral Density

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Concepts:

- Relation of Spectral Density to the Fourier Transform
  - Weiner-Khinchine Relationship
- Properties of Spectral Density
- Spectral Density and the Complex Frequency Plane
- Mean-Square Values From Spectral Density
- Relation of Spectral Density to the Autocorrelation Function
- White Noise, Black Noise, Pink Noise
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- Cross-Spectral Density
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Chapter 7: Spectral Density

Wiener–Khinchin Theorem

For WSS random processes, the autocorrelation function is time based and has a spectral decomposition given by the power spectral density.

We can define a power spectral density as the Fourier transform of the autocorrelation function:

$$S_{XX}(w) = \mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-i\omega \tau) \cdot d\tau$$

Based on this definition, we also have

$$S_{XX}(w) = \mathcal{F}[R_{XX}(\tau)] \quad R_{XX}(\tau) = \mathcal{F}^{-1}[S_{XX}(w)]$$

$$R_{XX}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(i\omega t) \cdot dw$$

Also see:

http://en.wikipedia.org/wiki/Wiener%E2%80%93Khinchin_theorem

Alternate textbook form,

This function is also defined as the spectral density function (or power-spectral density) and is defined for both f and w as:

$$S_{YY}(w) = \lim_{T \to \infty} \frac{E\left[|Y(w)|^2\right]}{2T} \quad \text{or} \quad S_{YY}(f) = \lim_{T \to \infty} \frac{E\left[|Y(f)|^2\right]}{2T}$$
Properties: The power spectral density as a function is always

- real,
- positive, (never negative as it is a magnitude)
- and an even function in w.

As an even function, the PSD may be expected to have a polynomial form as:

\[ S_{XX}(w) = S_0 \frac{w^{2n} + a_{2n-2}w^{2n-2} + a_{2n-4}w^{2n-4} + \cdots + a_2w^2 + a_0}{w^{2m} + b_{2m-2}w^{2m-2} + b_{2m-4}w^{2m-4} + \cdots + b_2w^2 + b_0} \]

where \( m > n \).

Finite property in frequency:

The Power Spectral Density must also approach zero as w approached infinity …. Therefore,

\[ S_{XX}(w \to \infty) = \lim_{w \to \infty} S_0 \frac{w^{2n} + a_{2n-2}w^{2n-2} + \cdots + a_2w^2 + a_0}{w^{2m} + b_{2m-2}w^{2m-2} + \cdots + b_2w^2 + b_0} \]

\[ \Rightarrow \lim_{w \to \infty} S_0 \frac{w^{2n}}{w^{2m}} = \lim_{w \to \infty} S_0 \frac{1}{w^{2(m-n)}} = 0 \]

For \( m > n \), the condition will be met.
**Generic Example of a Discrete Spectral Density (p. 267)**

\[ X(t) = A + B \cdot \sin(2\pi \cdot f_1 \cdot t + \theta_1) + C \cdot \cos(2\pi \cdot f_2 \cdot t + \theta_2) \]

where the phase angles are uniformly distributed R.V from 0 to 2\(\pi\).

With practice, we can see that

\[
R_{xx}(\tau) = A^2 + B^2 \cdot E \left[ \frac{1}{2} \cdot \cos(2\pi \cdot f_1 \cdot \tau) - \frac{1}{2} \cdot \cos(2\pi \cdot f_1 (2t + \tau) + 2\theta_1) \right] \\
+ C^2 \cdot E \left[ \frac{1}{2} \cdot \cos(2\pi \cdot f_2 \cdot \tau) + \frac{1}{2} \cdot \cos(2\pi \cdot f_2 (2t + \tau) + 2\theta_2) \right]
\]

which lead to

\[
R_{xx}(\tau) = A^2 + \frac{B^2}{2} \cdot \cos(2\pi \cdot f_1 \cdot \tau) + \frac{C^2}{2} \cdot \cos(2\pi \cdot f_2 \cdot \tau)
\]

And then taking the Fourier transform

\[
S_{xx}(f) = A^2 \cdot \delta(f) + \frac{B^2}{4} \cdot \left( \frac{1}{2} \cdot \delta(f + f_1) + \frac{1}{2} \cdot \delta(f - f_1) \right) + \frac{C^2}{4} \cdot \left( \delta(f + f_2) + \delta(f - f_2) \right)
\]

We also know from the before

\[
\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(w) \cdot dw = \int_{-\infty}^{\infty} S_{xx}(f) \cdot df
\]

Therefore, the 2nd moment can be immediately computed as

\[
\overline{X^2} = \int_{-\infty}^{\infty} \left[ A^2 \cdot \delta(f) + \frac{B^2}{4} \cdot \left( \delta(f + f_1) + \delta(f - f_1) \right) + \frac{C^2}{4} \cdot \left( \delta(f + f_2) + \delta(f - f_2) \right) \right] \cdot df
\]

\[
\overline{X^2} = A^2 + \frac{B^2}{4} \cdot (2) + \frac{C^2}{4} \cdot (2) = A^2 + \frac{B^2}{2} + \frac{C^2}{2}
\]

We can also see that

\[
\overline{X} = E\left[ A + B \cdot \sin(2\pi \cdot f_1 \cdot t + \theta_1) + C \cdot \cos(2\pi \cdot f_2 \cdot t + \theta_2) \right] = A
\]

So,

\[
\sigma^2 = A^2 + \frac{B^2}{2} + \frac{C^2}{2} - A^2 = \frac{B^2}{2} + \frac{C^2}{2}
\]
Another example: p. 267-269 more “periodic symbol transmission stuff”

Determine the autocorrelation of the binary sequence, assuming \( p = 0.5 \).

\[
x(t) = \sum_{k=-\infty}^{\infty} A_k \cdot p(t-t_0 - k \cdot T)
\]

\[
x(t) = p(t) \ast \sum_{k=-\infty}^{\infty} A_k \cdot \delta(t-t_0 - k \cdot T)
\]

Notice that there is a random part and a deterministic part that can be separated. Determine the auto correlation of the random discrete time sequence

\[
y(t) = \sum_{k=-\infty}^{\infty} A_k \cdot \delta(t-t_0 - k \cdot T)
\]

\[
E[y(t) \cdot y(t+\tau)] = E\left[ \left( \sum_{k=-\infty}^{\infty} A_k \cdot \delta(t-t_0 - k \cdot T) \right) \cdot \left( \sum_{j=-\infty}^{\infty} A_j \cdot \delta(t+\tau-t_0 - j \cdot T) \right) \right]
\]

\[
R_{yy}(\tau) = E[y(t) \cdot y(t+\tau)] = E\left[ \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A_j \cdot A_k \cdot \delta(t-t_0 - k \cdot T) \cdot \delta(t+\tau-t_0 - j \cdot T) \right]
\]

\[
R_{yy}(\tau) = \sum_{k=-\infty}^{\infty} E[A_k^2] \cdot E[\delta(t-t_0 - k \cdot T) \cdot \delta(t+\tau-t_0 - k \cdot T)]
\]

\[
+ \sum_{k=-\infty}^{\infty} \sum_{j=-\infty, j \neq k}^{\infty} E[A_j \cdot A_k] \cdot E[\delta(t-t_0 - k \cdot T) \cdot \delta(t+\tau-t_0 - j \cdot T)]
\]

Taking the time average of the expected values based on \( T \) periods,

\[
R_{yy}(\tau) = E[A_k^2] \cdot \frac{1}{T} \cdot \delta(\tau) + E[A_k]^2 \cdot \frac{1}{T} \cdot \sum_{m=-\infty}^{\infty} \delta(\tau-m \cdot T)
\]

\[
R_{yy}(\tau) = \left[ E[A_k^2] - E[A_k]^2 \right] \cdot \frac{1}{T} \cdot \delta(\tau) + E[A_k]^2 \cdot \frac{1}{T} \cdot \sum_{m=-\infty}^{\infty} \delta(\tau-m \cdot T)
\]

\[
R_{yy}(\tau) = \sigma_A^2 \cdot \frac{1}{T} \cdot \delta(\tau) + \mu_A^2 \cdot \frac{1}{T} \cdot \sum_{m=-\infty}^{\infty} \delta(\tau-m \cdot T)
\]

\[
S_{yy}(f) = \sigma_A^2 \cdot \frac{1}{T} + \mu_A^2 \cdot \frac{1}{T} \cdot \sum_{m=-\infty}^{\infty} \delta\left( f - \frac{m}{T} \right)
\]
From here, it can be shown that

\[ S_{xx}(f) = |P(f)|^2 \cdot S_{yy}(f) \]

\[ S_{xx}(f) = |P(f)|^2 \left[ \sigma_A^2 \cdot \frac{1}{T} + \mu_A^2 \cdot \frac{1}{T^2} \cdot \sum_{m=-\infty}^{\infty} \delta \left( f - \frac{m}{T} \right) \right] \]

\[ S_{xx}(f) = |P(f)|^2 \cdot \frac{\sigma_A^2}{T} + |P(f)|^2 \cdot \frac{\mu_A^2}{T^2} \cdot \sum_{m=-\infty}^{\infty} \delta \left( f - \frac{m}{T} \right) \]

This is a magnitude scaled version of the power spectral density of the pulse shape and numerous impulse responses with magnitudes shaped by the pulse at regular frequency intervals based on the signal periodicity.

The result was pictured in the textbook as …

A note on real signals, the “pulse” can be longer than the symbol period T. For symbol transmissions using “Square-Root Nyquist Filters” the pulse may actually be 3-10 symbols long with very specific properties so that “inter-symbol interference” will not occur when a correlation signal detector samples the received symbol at the “perfect moment in time”.

- see MRSP course example of exam #2 test signal.
Binary Pulse Amplitude (PAM) signaling formats

I did a series of similar definitions for the ECE4600 Communications course a few years ago … the results follow

(a) Unipolar RZ & NRZ , (b) Polar RZ & NRZ , (c) Bipolar NRZ , (d) Split-phase Manchester, and (e) Polar quaternary NRZ.

PAM Power Spectral Density: Polar NRZ

The random signal can be describes as

\[ v(t) = \sum_{k=-\infty}^{\infty} a_k \cdot \text{rect} \left( \frac{t-T_d-k \cdot T_b}{T_b} \right) \]

\[ p(T_d) = \frac{1}{T_b}, \quad 0 < T_d \leq T_b \quad \text{and} \quad E[a_n] = 0, E[a_n^2] = \sigma_a^2 \]

\[ R_{vv}(\tau) = E[v(t) \cdot v(t+\tau)] = \sigma_v^2 \cdot \left( 1 - \frac{|\tau|}{T_b} \right), \quad -T_b < \tau < T_b \]

\[ S_{vv}(w) = \Im \{ E[v(t) \cdot v(t+\tau)] \} = \sigma_v^2 \cdot T_b \cdot \text{sinc}^2 (f \cdot T_b) \]

PAM Power Spectral Density: Arbitrary Pulse – similar to our textbook

\[ v(t) = \sum_{k=-\infty}^{\infty} a_k \cdot P \left( \frac{t-T_d-k \cdot D}{D} \right) \]

\[ p(T_d) = \frac{1}{D}, \quad 0 < T_d \leq D \quad \text{and} \quad E[a_n] = m_a, E[a_n^2] = \sigma_a^2 + m_a^2 \]

\[ S_{vv}(f) = \frac{1}{D} \cdot |P(f)|^2 \cdot \sum_{n=-\infty}^{\infty} R_a(n) \cdot \exp(-j \cdot 2\pi \cdot f \cdot D) \]

\[ R_a(n) = \begin{cases} \sigma_a^2 + m_a^2, & n = 0 \\ m_a^2, & n \neq 0 \end{cases} \quad \text{and} \quad T_b = D, r_b = \frac{1}{D} \]

\[ S_{vv}(f) = \frac{\sigma_a^2}{D} \cdot |P(f)|^2 + \left( \frac{m_a}{D} \right)^2 \cdot \sum_{n=-\infty}^{\infty} \left| P \left( \frac{n}{D} \right) \right|^2 \cdot \delta \left( f - \frac{n}{D} \right) \]

\[ S_{vv}(f) = \sigma_a^2 \cdot r_b \cdot |P(f)|^2 + (m_a \cdot r_b)^2 \cdot \sum_{n=-\infty}^{\infty} \left| P \left( n \cdot r_b \right) \right|^2 \cdot \delta \left( f - n \cdot r_b \right) \]
Power spectrum of Unipolar, binary RZ signal

\[ p(t) = \text{rect} \left( \frac{t}{T_b/2} \right) = \text{rect}(2 \cdot r_b \cdot t) \quad \text{where} \quad P(f) = \frac{1}{2 \cdot r_b} \cdot \text{sinc} \left( \frac{f}{2 \cdot r_b} \right) \]

\[ E[a_n] = \frac{A}{2}, \quad E[a_n^2] = \frac{A^2}{2} \quad \text{and} \quad R_s(n) = \begin{cases} \sigma_a^2 + m_a^2 = \frac{A^2}{2}, & n = 0 \\ m_a^2 = \frac{A^2}{4}, & n \neq 0 \end{cases} \]

\[ S_v(f) = \frac{A^2}{16 \cdot r_b} \cdot \text{sinc} \left( \frac{f}{2 \cdot r_b} \right)^2 + \frac{A^2}{16} \cdot \sum_{n=-\infty}^{\infty} \text{sinc} \left( \frac{n}{2} \right)^2 \cdot \delta \left( f - n \cdot r_b \right) \]

Power spectrum of Unipolar, binary NRZ signal

\[ p(t) = \text{rect} \left( \frac{t}{T_b} \right) = \text{rect}(r_b \cdot t) \quad \text{where} \quad P(f) = \frac{1}{r_b} \cdot \text{sinc} \left( \frac{f}{r_b} \right) \]

\[ E[a_n] = \frac{A}{2}, \quad E[a_n^2] = \frac{A^2}{2} \quad \text{and} \quad R_s(n) = \begin{cases} \sigma_a^2 + m_a^2 = \frac{A^2}{2}, & n = 0 \\ m_a^2 = \frac{A^2}{4}, & n \neq 0 \end{cases} \]

\[ S_v(f) = \frac{A^2}{4 \cdot r_b} \cdot \text{sinc} \left( \frac{f}{r_b} \right)^2 + \frac{A^2}{4} \cdot \sum_{n=-\infty}^{\infty} \text{sinc} \left( n \right)^2 \cdot \delta \left( f - n \cdot r_b \right) \]

But based in the sinc function equals

\[ S_v(f) = \frac{A^2}{4 \cdot r_b} \cdot \text{sinc} \left( \frac{f}{r_b} \right)^2 + \frac{A^2}{4} \cdot \delta \left( f \right) \]
Power spectrum of Polar, binary RZ signal (+/- A/2)

\[ p(t) = \text{rect} \left( \frac{t}{\frac{T_b}{2}} \right) = \text{rect}(2 \cdot r_b \cdot t) \quad \text{where} \quad P(f) = \frac{1}{2 \cdot r_b} \cdot \text{sinc} \left( \frac{f}{2 \cdot r_b} \right) \]

\[ E[a_n] = 0, E[a_n^2] = \frac{A^2}{4} \quad \text{and} \quad R_a(n) = \begin{cases} \sigma_a^2 + m_a^2 = \frac{A^2}{4}, & n = 0 \\ m_a^2 = 0, & n \neq 0 \end{cases} \]

\[ S_{vv}(f) = \frac{A^2}{16 \cdot r_b^2} \cdot \text{sinc} \left( \frac{f}{2 \cdot r_b} \right)^2 \]

Power spectrum of Polar, binary NRZ signal (+/- A/2)

\[ p(t) = \text{rect} \left( \frac{t}{T_b} \right) = \text{rect}(r_b \cdot t) \quad \text{where} \quad P(f) = \frac{1}{r_b} \cdot \text{sinc} \left( \frac{f}{r_b} \right) \]

\[ E[a_n] = 0, E[a_n^2] = \frac{A^2}{4} \quad \text{and} \quad R_a(n) = \begin{cases} \sigma_a^2 + m_a^2 = \frac{A^2}{4}, & n = 0 \\ m_a^2 = 0, & n \neq 0 \end{cases} \]

\[ S_{vv}(f) = \frac{A^2}{4 \cdot r_b^2} \cdot \text{sinc} \left( \frac{f}{r_b} \right)^2 \]

Why do we care?

The bandwidth and spectral characteristics of the signals are very important. Issues include spectral capacity, filter selections, adjacent signal interference, etc.
Autocorrelation to Power Spectral Density (p. 283-284.)

Example 1: Random Pulse in Time - analysis 1

Let \( X(t) \) be an asynchronous bipolar signal with a random pulse width, \( T \), and amplitude +/-A. The pulse width, \( T \), is exponentially distributed.

\[
X(t) = A \cdot \text{rect} \left( \frac{t - \frac{T}{2} - t_0}{T} \right)
\]

Reminders for the exponential distribution

\[
f_T(t) = \begin{cases} 
\frac{1}{\mu_T} \cdot \exp \left( -\frac{1}{\mu_T} \cdot t \right) & t > 0 \\
0 & \text{else}
\end{cases}
\]

\[
E[T] = \int_0^{\infty} t \cdot \frac{1}{\mu_T} \cdot \exp \left( -\frac{1}{\mu_T} \cdot t \right) \cdot dt = \frac{1}{\mu_T} \cdot \frac{1}{\left( \frac{1}{\mu_T} \right)^2} \cdot \exp \left( -\frac{1}{\mu_T} \cdot t \right) \cdot \left( -\frac{1}{\mu_T} \cdot t - 1 \right) \bigg|_0^{\infty}
\]

\[
E[T] = -\mu_T \cdot \exp \left( -\frac{1}{\mu_T} \cdot 0 \right) \cdot \left( -\frac{1}{\mu_T} \cdot 0 - 1 \right) = -\mu_T \cdot 1 \cdot (-1) = \mu_T
\]

\[
E[T^2] = \int_0^{\infty} t^2 \cdot \frac{1}{\mu_T} \cdot \exp \left( -\frac{1}{\mu_T} \cdot t \right) \cdot dt
\]

\[
E[T^2] = \left. \frac{1}{\mu_T} \cdot \frac{1}{\left( -\frac{1}{\mu_T} \right)^3} \cdot \exp \left( -\frac{1}{\mu_T} \cdot t \right) \cdot \left( \frac{1}{\mu_T} \cdot t^2 + 2 \cdot \frac{1}{\mu_T} \cdot t + 2 \right) \right|_0^{\infty}
\]

\[
E[T^2] = -\left( -\mu_T^2 \right) \cdot \exp \left( -\frac{1}{\mu_T} \cdot 0 \right) \cdot (2) = 2 \cdot \mu_T^2
\]

\[
\sigma_T^2 = 2 \cdot \mu_T^2 - \left( \mu_T \right)^2 = \mu_T^2
\]

And returning to the autocorrelation …
Now for the autocorrelation:

\[
R_{XX}(t, t + \tau) = E[X(t) \cdot X(t + \tau)] = E \left[ A \cdot \text{rect} \left( \frac{t - T - t_0}{T} \right) \cdot A \cdot \text{rect} \left( \frac{t + \tau - T - t_0}{T} \right) \right]
\]

\[
R_{XX}(t, t + \tau) = E[A^2] \cdot E \left[ \text{rect} \left( \frac{t - T - t_0}{T} \right) \cdot \text{rect} \left( \frac{t + \tau - T - t_0}{T} \right) \right]
\]

For \( \tau > 0 \) we have

\[
R_{XX}(\tau) = E[A^2] \cdot \Pr[\tau < T] = E[A^2] \cdot \int_{\tau}^{\infty} \int_{\tau}^{\infty} 1 \cdot 1 \cdot \frac{1}{\mu_r} \cdot \exp \left( -\frac{1}{\mu_r} \cdot x \right) \cdot dx
\]

\[
R_{XX}(\tau) = E[A^2] \cdot \left[ \frac{1}{\mu_r} \cdot \exp \left( -\frac{1}{\mu_r} \cdot \tau \right) \right]_{\tau}^{\infty}
\]

\[
R_{XX}(\tau) = A^2 \cdot \exp \left( -\frac{1}{\mu_r} \cdot \tau \right)
\]

For a real symmetric autocorrelation, the negative portion must be correctly included as

\[
R_{XX}(\tau) = A^2 \cdot \exp \left( -\frac{1}{\mu_r} \cdot |\tau| \right)
\]

The power spectral density based on the auto-correlation becomes ….
And the power spectral density becomes

\[
S_{XX}(w) = \mathbb{E}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-iw\tau) \cdot d\tau
\]

\[
S_{XX}(w) = \int_{-\infty}^{\infty} A^2 \cdot \exp \left( -\frac{1}{\mu_T} \cdot |\tau| \right) \cdot \exp(-iw\tau) \cdot d\tau
\]

\[
S_{XX}(w) = A^2 \cdot \left[ \int_{0}^{\infty} \exp \left( -\frac{1}{\mu_T} \cdot i\omega \cdot \tau \right) \cdot d\tau + \int_{-\infty}^{0} \exp \left( \frac{1}{\mu_T} \cdot i\omega \cdot \tau \right) \cdot d\tau \right]
\]

\[
S_{XX}(w) = A^2 \cdot \left[ \int_{0}^{\infty} \frac{1}{\mu_T} \cdot \exp \left( -\frac{1}{\mu_T} \cdot i\omega \cdot \tau \right) \cdot d\tau + \int_{-\infty}^{0} \frac{1}{\mu_T} \cdot \exp \left( \frac{1}{\mu_T} \cdot i\omega \cdot \tau \right) \cdot d\tau \right]
\]

\[
S_{XX}(w) = A^2 \cdot \left[ -\frac{1}{\mu_T} \cdot \exp \left( -\frac{1}{\mu_T} \cdot i\omega \cdot 0 \right) + \frac{1}{\mu_T} \cdot \exp \left( \frac{1}{\mu_T} \cdot i\omega \cdot 0 \right) \right] = A^2 \cdot \left[ \frac{1}{\mu_T} \cdot \exp \left( \frac{1}{\mu_T} \cdot i\omega \cdot 0 \right) \right]
\]

\[
S_{XX}(w) = A^2 \cdot \frac{2}{\mu_T} \cdot \frac{1}{\mu_T} + w^2
\]

For graphical example of these functions, see Figure 7-8 in the text.

![Figure 7-8](image)

**Figure 7-8** Relation between (a) autocorrelation function and (b) spectral density.

One of the favorite signals used as an example in the textbook ....
Example 2 Bipolar signal with a random pulse-width – uniform dist.

Let X(t) be an asynchronous bipolar signal with a random pulse width, T, and amplitude +/-A. The pulse width uniformly distributed.

\[
X(t) = \sum_k A_k \cdot \text{Rect}\left(\frac{t - t_0 + \frac{T_W}{2} + kT}{T_W}\right)
\]

where A(k) are zero mean, identically distributed, independent R.V., t0 is a R.V. uniformly distributed over possible bit period T, and Tw is a random variable uniformly distributed from 0 to T. k describes the bit positions.

For the autocorrelation of one of the pulse periods:

\[
R_{XX}(t, t + \tau) = E\left[A^2 \cdot \text{Rect}\left(\frac{t - t_0 + \frac{T_W}{2} + kT}{T_W}\right) \cdot \text{Rect}\left(\frac{t + \tau - t_0 + \frac{T_W}{2} + kT}{T_W}\right)\right]
\]

\[
R_{XX}(t, t + \tau) = E\left[A^2 \cdot \text{Rect}\left(\frac{t - t_0 + \frac{T_W}{2} + kT}{T_W}\right) \cdot \text{Rect}\left(\frac{t + \tau - t_0 + \frac{T_W}{2} + kT}{T_W}\right)\right]
\]

For \(\tau > 0\) we have

\[
R_{XX}(\tau) = E\left[A^2 \cdot \Pr[\tau < T_w]\right] = E\left[A^2 \right] \int_0^\tau \frac{1}{T} \cdot dx
\]

\[
R_{XX}(\tau) = E\left[A^2 \right] \cdot \left[\frac{x}{T}\right]^T_\tau
\]

\[
R_{XX}(\tau) = A^2 \cdot \left(\frac{T - \tau}{T}\right)
\]

For a real symmetric autocorrelation, the negative portion must be correctly included as

\[
R_{XX}(\tau) = A^2 \cdot \left(1 - \frac{\left|\tau\right|}{T}\right)
\]
And the power spectral density becomes

$$S_{XX}(w) = \mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-iw\tau) \cdot d\tau$$

$$S_{XX}(w) = \int_{-\infty}^{\infty} A^2 \cdot \left(1 - \frac{|\tau|}{T}\right) \cdot \exp(-iw\tau) \cdot d\tau$$

$$S_{XX}(w) = A^2 \cdot \left[\int_{0}^{T} \left(1 - \frac{\tau}{T}\right) \cdot \exp(-iw\tau) \cdot d\tau + \int_{-T}^{0} \left(1 + \frac{\tau}{T}\right) \cdot \exp(-iw\tau) \cdot d\tau\right]$$

$$S_{XX}(w) = A^2 \cdot \left[\left(\frac{1}{-iw} - \frac{1}{T} \cdot \frac{-iw \cdot \tau - 1}{(-iw)^2}\right) \cdot \exp(-iw \cdot T) - \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(-iw \cdot \tau)\right]_{0}^{T} + \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(-iw \cdot \tau)$$

$$S_{XX}(w) = A^2 \cdot \left[\left(\frac{1}{-iw} - \frac{1}{T} \cdot \frac{-iw \cdot T - 1}{(-iw)^2}\right) \cdot \exp(-iw \cdot T) - \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(-iw \cdot \tau)\right] + \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(iw \cdot T)$$

$$S_{XX}(w) = A^2 \cdot \left[\left(\frac{1}{-iw} - \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(-iw \cdot T) - \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(iw \cdot T)\right]$$

$$S_{XX}(w) = A^2 \cdot \left[\left(\frac{1}{-iw} + \frac{1}{w^2 T}\right) \cdot \exp(-iw \cdot T) - \left(\frac{1}{-iw} + \frac{1}{T} \cdot \frac{-1}{(-iw)^2}\right) \cdot \exp(iw \cdot T)\right]$$

$$S_{XX}(w) = A^2 \cdot \left[\frac{1}{w^2 T} \cdot \exp\left(-iw \cdot \frac{T}{2}\right) - \exp\left(iw \cdot \frac{T}{2}\right)\right]^2$$

$$S_{XX}(w) = A^2 \cdot \frac{-1}{w^2 T} \cdot \left[-2i \cdot \sin\left(w \cdot \frac{T}{2}\right)\right]^2 = A^2 \cdot \frac{4}{w^2 T} \cdot \sin^2\left(w \cdot \frac{T}{2}\right)$$

$$S_{XX}(w) = A^2 \cdot \frac{\sin^2\left(w \cdot \frac{T}{2}\right)}{\left(w \cdot \frac{T}{2}\right)^2} = A^2 \cdot \frac{\sin^2\left(2\pi f \cdot \frac{T}{2}\right)}{\left(2\pi f \cdot \frac{T}{2}\right)^2} = A^2 \cdot T \cdot \text{sinc}^2(f \cdot T)$$

Note: \(\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}\) this follows Matlab’s definitions of the sinc function.
Note: what happens if the amplitude is not a binary value between +/- A.

(1) If it remains zero mean,

The magnitude of the autocorrelation and power spectral density is multiplied by the second moment of A.

(2) If it is not zero mean …

The cross-correlation of bits to other bits is no longer zero, as

\[ E[A_i \cdot A_k] = E[A_i] \cdot E[A_k] \neq 0 \]

to make a long story short, the cross-correlation operation is identical to that shown for the autocorrelation at periodic locations in time tau, except that the “amplitude” is now the square of the mean instead of the second moment!

From chapter 6 …. for T rect functions

\[ \mathfrak{R}_{XX}(\tau) = E[A^2] \cdot \left[ 1 - \frac{|\tau|}{T} \right] + E[A]^2 \]

\[ S_{XX}(w) = \mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-iw\tau) \cdot d\tau \]

\[ S_{XX}(w) = \sigma_A^2 \cdot T \cdot \text{sinc}(f \cdot T)^2 + \mu_A^2 \cdot \delta(f) \]

From chapter 6 …. for \( \alpha T < T \) rect functions

\[ \mathfrak{R}_{XX}(\tau) = E[A^2] \cdot \alpha \cdot \left[ 1 - \frac{|\tau|}{\alpha T} \right] + E[A]^2 \cdot \alpha \cdot \left[ 1 - \frac{\tau \pm n \cdot T}{\alpha T} \right] \]

Then, the power spectral density consists of the Fourier Transform of the superposition of two functions, (1) periodic triangles of magnitude \( E[A]^2 \) and (2) a single triangle at the origin of magnitude \( \sigma_A^2 = E[A^2] - E[A]^2 \).

Note that the periodic triangle is equivalent to the convolution of a single triangle function with a comb function in time with spacing T. Thus, the PSD consists of a continuous sinc^2 function (2 above) summed with a “line-spectrum” sinc^2 function (1 above). This is similar to figure 7-3!
Deriving the Mean-Square Values from the Power Spectral Density

Using the Fourier transform relation between the Autocorrelation and PSD

\[
S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-iw\tau) \cdot d\tau
\]

\[
R_{XX}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(iwt) \cdot dw
\]

The mean squared value of a random process is equal to the 0th lag of the autocorrelation

\[
E[X^2] = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(iw \cdot 0) \cdot dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot dw
\]

\[
E[X^2] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f) \cdot \exp(i2\pi f \cdot 0) \cdot df = \int_{-\infty}^{\infty} S_{XX}(f) \cdot df
\]

Therefore, to find the second moment, integrate the PSD over all frequencies.

As a note, since the PSD is real and symmetric, the integral can be performed as

\[
E[X^2] = R_{XX}(0) = 2 \cdot \frac{1}{2\pi} \int_{0}^{\infty} S_{XX}(w) \cdot dw
\]

\[
E[X^2] = R_{XX}(0) = 2 \int_{0}^{\infty} S_{XX}(f) \cdot df
\]
Converting between Autocorrelation and Power Spectral Density

Using the properties of the functions

The power spectral density as a function is always

- real,
- positive,
- and an even function in w/f.

You can convert between the domains using:

The Fourier Transform in w

\[ S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-i\omega\tau) \cdot d\tau \]

\[ R_{XX}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(i\omega t) \cdot dw \]

The Fourier Transform in f

\[ S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-i2\pi f\tau) \cdot d\tau \]

\[ R_{XX}(t) = \int_{-\infty}^{\infty} S_{XX}(f) \cdot \exp(i2\pi ft) \cdot df \]

The 2-sided Laplace Transform

\[ S_{XX}(s) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-s\tau) \cdot d\tau \]

\[ R_{XX}(t) = \frac{1}{j2\pi} \int_{-j\infty}^{j\infty} S_{XX}(s) \cdot \exp(st) \cdot ds \]
Notes on using the Laplace Transform

(1) When converting from the s-domain to the frequency domain use:

\[ s = jw \quad \text{or} \quad w = -js \]

(2) As an even function, the PSD may be expected to have a polynomial form as:

\[ S_{XX}(w) = S_0 \frac{w^{2n} + a_{2n-2}w^{2n-2} + a_{2n-4}w^{2n-4} + \cdots + a_2w^2 + a_0}{w^{2m} + b_{2m-2}w^{2m-2} + b_{2m-4}w^{2m-4} + \cdots + b_2w^2 + b_0} \]

This can be expressed as:

\[ S_{XX}(w) = \frac{c(s) \cdot c(-s)}{d(s) \cdot d(-s)} \]

To compute the autocorrelation function for \( 0 \leq \tau \) use a partial fraction expansion such that

\[ S_{XX}(w) = \frac{g(s)}{d(s)} + \frac{g(-s)}{d(-s)} \]

and solve for

\[ R_{XX}(\tau) = \frac{1}{2\pi} \int_{-j\infty}^{j\infty} \frac{g(s)}{d(s)} \cdot \exp(st) \cdot ds \]

for determining \( 0 \geq \tau \), use the RHP expansion, replace \(-s\) with \(s\), perform the Laplace transform and replace \(t\) with \(-t\).
Example:

\[ S_{XX}(w) = \frac{A^2 \cdot \frac{2}{\mu_X}}{ \left( \frac{1}{\mu_X} \right)^2 + w^2} = \frac{2 \cdot A^2 \cdot \beta}{\beta^2 + w^2} \]

Substitute for \( w \)

\[ S_{XX}(s) = \frac{2 \cdot A^2 \cdot \beta}{\beta^2 - s^2} = \frac{2 \cdot A^2 \cdot \beta}{(\beta + s) \cdot (\beta - s)} \]

Partial fraction expansion

\[ S_{XX}(s) = \frac{k_0}{(\beta + s)} + \frac{k_1}{(\beta - s)} = \frac{k_0 \cdot (\beta - s) + k_1 \cdot (\beta + s)}{(\beta + s) \cdot (\beta - s)} = \frac{2 \cdot A^2 \cdot \beta}{(\beta + s) \cdot (\beta - s)} \]

\((-k_0 + k_1) \cdot s = 0 \implies k_0 = k_1\]

\((k_0 + k_1) \cdot \beta = 2 \cdot A^2 \cdot \beta \implies 2k_0 = 2 \cdot A^2\]

\[ S_{XX}(s) = \frac{A^2}{(\beta + s)} + \frac{A^2}{(\beta - s)} \]

Taking the LHP Laplace Transform

\[ L \left[ \frac{A^2}{(\beta + s)} \right] = A \cdot \exp(-\beta t) \quad \text{for } t > 0 \]

Taking the RHP with \(-s\) and then \(-t\).

\[ L \left[ \frac{A^2}{(\beta - (-s))} \right] = A^2 \cdot \exp(-\beta t) \implies A^2 \cdot \exp(-\beta (-t)) = A^2 \cdot \exp(\beta t) \quad \text{for } t < 0 \]

Combining we have

\[ R_{XX}(\tau) = A^2 \cdot \exp(-\beta \cdot |\tau|) \]

White Noise

Noise is inherently defined as a random process.

You may be familiar with “thermal” noise, based on the energy of an atom and the mean-free path that it can travel.

As a random process, whenever “white noise” is measured, the values are uncorrelated with each other, not matter how close together the samples are taken in time.

Further, we envision “white noise” as containing all spectral content, with no explicit peaks or valleys in the power spectral density.

As a result, we define “White Noise” as

\[ R_{XX}(\tau) = S_0 \cdot \delta(\tau) \]

\[ S_{XX}(w) = S_0 \]

This is an approximation or simplification because the area of the power spectral density is infinite!

Nominally, noise is defined within a bandwidth to describe the power. For example,

Thermal noise at the input of a receiver is defined in terms of kT, Boltzmann’s constant times absolute temperature, in terms of Watts/Hz. Thus there is kT Watts of noise power in every Hz of bandwidth.

For communications, this is equivalent to –174 dBm/Hz or –144 dBW/Hz.

For typical applications, we are interested in Band-Limited White Noise where

\[ S_{XX}(w) = \begin{cases} S_0 & |f| \leq W \\ 0 & W < |f| \end{cases} \]

The equivalent noise power is then:

\[ E[X^2] = R_{XX}(0) = \int_{-W}^{W} S_0 \cdot dw = 2 \cdot W \cdot S_0 \]

For communications, we use kTB (yes, we are off by a factor of 2 based on +/- freq).

How much noise power, in dBm, would I say that there is in a 1 MHz bandwidth?

\[ dB(kTB) \Rightarrow dB(kT) + dB(B) \Rightarrow -174 + 60 = -114 dBm \]
**Receiver Sensitivity**

What does it mean when you buy a receiver?

For a great receiver (spectrum analyzer grade), assume a 200 kHz FM radio bandwidth.

<table>
<thead>
<tr>
<th>Noise Power</th>
<th>kT</th>
<th>-174. dBm/Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent Noise Bandwidth</td>
<td>B</td>
<td>53. dB Hz</td>
</tr>
<tr>
<td>Receiver Noise Figure</td>
<td>NF</td>
<td>10. dB</td>
</tr>
<tr>
<td>Signal Detection Threshold</td>
<td>D</td>
<td>8. dB</td>
</tr>
</tbody>
</table>

Minimum Detectable Signal | MDS | -103. dBm

FM radio stations can transmit up to 1 Megawatt $\rightarrow$ +90 dBm

Why doesn’t your receiver get blasted? Path loss, distance, higher noise figure, receiving antenna inefficiency, etc.
Band Limited White Noise

\[ S_{XX}(w) = \begin{cases} S_0 & |f| \leq W \\ 0 & W < |f| \end{cases} \]

The equivalent noise power is then:

\[ E[X^2] = R_{XX}(0) = \int_{-W}^{W} S_0 \cdot dw = 2 \cdot W \cdot S_0 \]

but what about the autocorrelation?

\[ R_{XX}(t) = \int_{-W}^{W} S_0 \cdot \exp(i2\pi ft) \cdot df \]

\[ R_{XX}(t) = S_0 \cdot \left[ \frac{\exp(i2\pi ft)}{i2\pi} \right]_{-W}^{W} = S_0 \cdot \left[ \frac{\exp(i2\pi Wt)}{i2\pi} - \frac{\exp(-i2\pi Wt)}{i2\pi} \right] \]

\[ R_{XX}(t) = S_0 \cdot \frac{2 \cdot i \cdot \sin(i2\pi Wt)}{i2\pi} \]

For \( \sin(\pi t) = \frac{\sin(\pi t)}{\pi t} \)

\[ R_{XX}(t) = 2 \cdot W \cdot S_0 \cdot \text{sinc}(2Wt) \]

Using the concept of correlation, for what values will the autocorrelation be zero? (At these delays in time, sampled data would be uncorrelated with previous samples!)

\[ 2Wt = k \]

\[ t = \frac{k}{2W} \quad \text{for} \ k = \pm 1, \pm 2, \cdots \]

Sampling at 1/2W seems to be a good idea, but isn’t that the Nyquist rate!!

Also note, noise passed through a filter becomes band-limited, and the narrower the filter the smaller the noise power … but the wider is the sinc autocorrelation function.
The Cross-Spectral Density

Why not form the power spectral response of the cross-correlation function?

The Fourier Transform in $w$

$$S_{XY}(w)=\int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \exp(-i\omega\tau) \cdot d\tau \quad \text{and} \quad S_{YX}(w)=\int_{-\infty}^{\infty} R_{YX}(\tau) \cdot \exp(-i\omega\tau) \cdot d\tau$$

$$R_{XY}(t)=\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(w) \cdot \exp(i\omega t) \cdot dw \quad \text{and} \quad R_{YX}(t)=\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(w) \cdot \exp(i\omega t) \cdot dw$$

Properties of the functions

$$S_{XY}(w)=\text{conj}(S_{YX}(w))$$

Since the cross-correlation is real,

- the real portion of the spectrum is even
- the imaginary portion of the spectrum is odd

There are no other important (assumed) properties to describe

Note: the trick using the Laplace transform to form the positive and negative portions of the “time-based” cross-correlation is required to determine the correct “inverse transform”.