Chapter 7: Spectral Density

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Concepts:

- Relation of Spectral Density to the Fourier Transform
  - Weiner-Khinchine Relationship
- Properties of Spectral Density
- Spectral Density and the Complex Frequency Plane
- Mean-Square Values From Spectral Density
- Relation of Spectral Density to the Autocorrelation Function
- White Noise, Black Noise, Pink Noise
- Contour Integration – (Appendix I)
- Cross-Spectral Density
- Autocorrelation Function Estimate of Spectral Density
- Periodogram Estimate of Spectral Density

Wiener–Khinchin Theorem
For WSS random processes, the autocorrelation function is time based and has a spectral decomposition given by the power spectral density.

Also see:

http://en.wikipedia.org/wiki/Wiener%E2%80%93Khinchin_theorem
Chapter 7: Spectral Density

Periodograms and Windowing Data

We can window the data prior to autocorrelation.

- this will cause a change in the energy that must be compensated

\[
\text{AvgEnergy} = \frac{1}{T} \int_0^T [x(t) \cdot w(t)]^2 \cdot dt
\]

\[
E[\text{AvgEnergy}] = \frac{1}{T} \int_0^T [E[x(t)^2] \cdot w(t)^2] \cdot dt = E[x(t)^2] \cdot \frac{1}{T} \int_0^T w(t)^2 \cdot dt
\]

For a rectangular window, this becomes

\[
E[\text{AvgEnergy}] = E[x(t)^2]
\]

If an alternate window is used, a power scaling will be necessary to maintain the same estimated energy.

For a given window, function we can apply:

\[
scale = \sqrt{\frac{1}{T} \int_0^T w(t)^2 \cdot dt}
\]

So that

\[
w_{scale}(t) = \frac{w(t)}{scale} = w(t) / \sqrt{\frac{1}{T} \int_0^T w(t)^2 \cdot dt}
\]

This also applies to the sampled data computations where

\[
w_{scale}(n) = \frac{w(n)}{scale} = w(n) / \sqrt{\frac{1}{N} \sum_{m=0}^{N-1} w(m)^2}
\]

Now, multiple smaller results may be processed to generate an average.

The spectral resolution (number of frequency bins) necessarily decreases, but now multiple spectra and can be averaged together based on “unique” time sample sets of the data.

It is useful for observing time varying phenomena and capturing the values when the desired signal is present.

See WaterFallDemo.m for a spectral periodogram ….
MATLAB & Homework Examples from Chapter 6 & 7

Note: Some homework problems to review … in class 7-5.1 and 7-6.3

7-5.1 Based on Problem 7-3.1.

a) Find the mean-square value of the random process in Problem 7-3.1(b). Doing 7-3.1(a) is does not make sense and the solution manual is wrong …

\[ S_{xx}(w) = \frac{w^2 + 16}{w^4 + 9 \cdot w^2 + 18} \]

\[ S_{xx}(s) = \frac{-s^2 + 16}{s^4 - 9 \cdot s^2 + 18} = \frac{-s^2 + 16}{(s^2 - 6) \cdot (s^2 - 3)} = \frac{-(s - 4) \cdot (s + 4)}{(s - \sqrt{6}) \cdot (s + \sqrt{6}) \cdot (s - \sqrt{3}) \cdot (s + \sqrt{3})} \]

\[ S_{xx}(s) = \frac{-(s - 4) \cdot (s + 4)}{(s - \sqrt{6}) \cdot (s + \sqrt{3}) \cdot (s + \sqrt{6})} \]

where \[ c(s) = \frac{s + 4}{s + \sqrt{6} \cdot s + \sqrt{3}} = \frac{s + 4}{s^2 + (\sqrt{6} + \sqrt{3}) \cdot s + 3 \cdot \sqrt{2}} \]

Using Table 7-1

\[ I_2 = \frac{c_0^2 \cdot d_0 + c_1^2 \cdot d_2}{2 \cdot d_0 \cdot d_1 \cdot d_2} = \frac{1^2 \cdot 3 \cdot \sqrt{2} + 4^2 \cdot 1}{2 \cdot 1 \cdot (\sqrt{6} + \sqrt{3}) \cdot 3 \cdot \sqrt{2}} \]

\[ I_2 = \frac{3 \cdot \sqrt{2} + 16}{6 \cdot \sqrt{12} + 6 \cdot \sqrt{6}} = \frac{20.24}{35.48} = 0.571 \]

Repeat part (a) using contour integration in the complex frequency plane.

The LHP poles are \(-\sqrt{6}\) and \(-\sqrt{3}\). Using the residue computations for the magnitude factors (as shown on p. 278-279).

\[ K_{-\sqrt{6}} = \left[ (s + \sqrt{6}) \cdot S_{xx}(s) \right]_{s = -\sqrt{6}} = \left. \frac{-(s - 4) \cdot (s + 4)}{(s - \sqrt{6}) \cdot (s - \sqrt{3}) \cdot (s + \sqrt{3})} \right|_{s = -\sqrt{6}} \]

\[ K_{-\sqrt{6}} = \frac{-(\sqrt{6} - 4) \cdot (\sqrt{6} + 4)}{(-\sqrt{6} - \sqrt{6}) \cdot (\sqrt{6} - \sqrt{3}) \cdot (\sqrt{6} + \sqrt{3})} \]

\[ \frac{K_{-\sqrt{6}} = \frac{10}{-2 \cdot \sqrt{6} \cdot (6 - 3)} = \frac{-5}{3 \cdot \sqrt{6}}} \]
\[
K_{-\sqrt{3}} = \left[s + \sqrt{3}\right] \cdot S_{XX}(s) \bigg|_{s = -\sqrt{3}} = \frac{-(s-4)(s+4)}{(s-\sqrt{3})(s+\sqrt{6})(s+\sqrt{6})} \\
K_{-\sqrt{3}} = \frac{\left(-\sqrt{3} - 4\right)\left(-\sqrt{3} + 4\right)}{\left(-\sqrt{3} - \sqrt{3}\right)\left(-\sqrt{3} - \sqrt{6}\right)\left(-\sqrt{3} + \sqrt{6}\right)} \\
K_{-\sqrt{3}} = \frac{13}{-2 \cdot \sqrt{3}(3-6)} = \frac{13}{6 \cdot \sqrt{3}} \\
E[X(t)^2] = K_{-\sqrt{3}} + K_{-\sqrt{6}} = \frac{13}{6 \cdot \sqrt{3}} - \frac{-5}{3 \cdot \sqrt{6}} = 1.251 - 0.680 = 0.571
\]

Repeat part (a) using direct integration.

\[
S_{XX}(w) = \frac{w^2 + 16}{w^4 + 9 \cdot w^2 + 18} \\
R_{XX}(0) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} \frac{w^2 + 16}{w^4 + 9 \cdot w^2 + 18} \cdot dw \\
R_{XX}(0) = \frac{1}{2 \cdot \pi} \cdot \frac{1}{3} \cdot \left[ -10 \cdot \frac{\sqrt{6}}{\sqrt{6}} \cdot \arctan\left(\frac{w}{\sqrt{6}}\right)_{\infty} + 13 \cdot \frac{\sqrt{3}}{\sqrt{3}} \cdot \arctan\left(\frac{w}{\sqrt{3}}\right)_{\infty} \right] \\
R_{XX}(0) = \frac{1}{2 \cdot \pi} \cdot \frac{1}{3} \cdot \left[ -10 \cdot \frac{\pi}{2} - \frac{\pi}{2} + 13 \cdot \frac{\sqrt{3}}{\sqrt{2}} \cdot \arctan\left(\frac{\pi}{2} - \frac{\pi}{2}\right) \right] \\
R_{XX}(0) = \frac{1}{2 \cdot \pi} \cdot \frac{1}{3} \cdot \left[ \frac{13 \cdot \pi}{\sqrt{3}} - \frac{10 \cdot \pi}{\sqrt{6}} \right] = \frac{1}{6} \cdot \left[ \frac{13 \cdot \sqrt{3}}{\sqrt{6}} - \frac{10 \cdot \sqrt{6}}{\sqrt{6}} \right] = 0.571
\]
A stationary random process has a spectral density of:

\[ S_{xx}(w) = \begin{cases} 5, & 10 \leq |w| \leq 20 \\ 0, & \text{else} \end{cases} \]

(a) Find the mean-square value of the process.

\[
R_{xx}(0) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} S_{xx}(w) \cdot dw = \frac{2}{2 \cdot \pi} \cdot \int_{0}^{\infty} S_{xx}(w) \cdot dw
\]

\[
R_{xx}(0) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{20} 5 \cdot dw + \frac{1}{2 \cdot \pi} \cdot \int_{-20}^{10} 5 \cdot dw = \frac{2}{2 \cdot \pi} \cdot \int_{10}^{20} 5 \cdot dw
\]

\[
R_{xx}(0) = \frac{10}{2 \cdot \pi} \cdot 5 \cdot \frac{20}{10} = \frac{10}{2 \cdot \pi} \cdot (20 - 10) = \frac{50}{\pi}
\]

(b) Find the auto-correlation function the process.

\[
R_{xx}(t) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} S_{xx}(w) \cdot \exp(j \cdot w \cdot t) \cdot dw
\]

\[
R_{xx}(t) = \frac{5}{2 \cdot \pi} \cdot \left( \int_{10}^{20} \exp(j \cdot w \cdot t) \cdot dw + \int_{-20}^{10} \exp(j \cdot w \cdot t) \cdot dw \right)
\]

\[
R_{xx}(t) = \frac{5}{2 \cdot \pi} \cdot \left( \exp(j \cdot w \cdot t) \bigg|_{10}^{20} \exp(j \cdot w \cdot t) \bigg|_{-20}^{10} \right)
\]

\[
R_{xx}(t) = \frac{5}{2 \cdot \pi} \cdot \left( \frac{\exp(j \cdot 20 \cdot t)}{j \cdot t} - \frac{\exp(j \cdot 10 \cdot t)}{j \cdot t} + \frac{\exp(j \cdot -10 \cdot t)}{j \cdot t} - \frac{\exp(j \cdot -20 \cdot t)}{j \cdot t} \right)
\]

\[
R_{xx}(t) = \frac{5}{2 \cdot \pi} \cdot \left( \frac{\exp(j \cdot 20 \cdot t)}{j \cdot t} - \frac{\exp(-j \cdot 20 \cdot t)}{j \cdot t} \right) - \left[ \frac{\exp(j \cdot 10 \cdot t)}{j \cdot t} - \frac{\exp(-j \cdot 10 \cdot t)}{j \cdot t} \right]
\]
\[ R_{XX}(t) = \frac{5}{2 \cdot \pi} \left( \frac{2 \cdot j \cdot \sin(20 \cdot t)}{j \cdot t} - \frac{2 \cdot j \cdot \sin(10 \cdot t)}{j \cdot t} \right) = \frac{5}{\pi \cdot t} \cdot (\sin(20 \cdot t) - \sin(10 \cdot t)) \]

\[ R_{XX}(t) = \frac{5}{\pi \cdot t} \left( 2 \cdot \sin \left( \frac{20 - 10}{2} \cdot t \right) \cdot \cos \left( \frac{20 + 10}{2} \cdot t \right) \right) = \frac{10}{\pi \cdot t} \cdot \sin(5 \cdot t) \cdot \cos(15 \cdot t) \]

\[ R_{XX}(t) = \frac{50}{\pi} \cdot \frac{\sin(5 \cdot t)}{5 \cdot t} \cdot \cos(15 \cdot t) = \frac{50}{\pi} \cdot \frac{\sin(5 \cdot t)}{\pi} \cdot \cos(15 \cdot t) \]

(c) Find the value of the auto-correlation function at \( t=0 \).

\[ R_{XX}(0) = \frac{50}{\pi} \cdot \frac{\sin(5 \cdot 0)}{5 \cdot 0} \cdot \cos(15 \cdot 0) = \frac{50}{\pi} \cdot \frac{\sin(5 \cdot 0)}{\pi} \cdot \cos(15 \cdot 0) \]

\[ R_{XX}(0) = \frac{50}{\pi} \cdot (1) \cdot (1) = \frac{50}{\pi} \cdot (1) \cdot (1) \]

\[ R_{XX}(0) = \frac{50}{\pi} \]

It must produce the same result!
Matlab from the textbook

From p. 223: corb

As previously mentioned …

The mathematics require correlation as compared to convolution.

Correlation

\[ y(\tau) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(\tau + \lambda) \cdot d\lambda \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n + k) \]

Convolution

\[ y(\tau) = \int_{-\infty}^{\infty} h(\lambda) \cdot x(\tau - \lambda) \cdot d\lambda \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n - k) \]

Something is “flipped” to achieve the other function. Note that the function scales by \(1/N+1\)

```matlab
function [ndt,R] = corb(a,b,f)
%
% corb.m biased correlation function
% a, b are equal length sampled time functions
% f is the sampling frequency
% ndt is the lag value for \pm time delays
%
% Cooper and McGillem
% p. 223

N = length(a);
M = length(b);

if (N~=M)
    error('corb: a and b must be equal in length.');
end

% Convert a and b to column vectors
aa=a(:);
bb=b(:);

% additional flipud added in case b is a column vector
R = conv(aa,flipud(bb))/(N+1);
ndt = (-(N-1):(N-1))/f;

if size(a,1)==1
    R=R';
end
```

This leads to the examples: Corxmp1, Corxmp2, Corxmp3 and Corxmp4

```matlab
%%
clear; close all;
rand('seed',1000);  % use seed to make repeatable
x=10*randn(1,1001);  % generate random samples
t1=0:0.001:1;  % sampling index
[t,R]=corb(x,x,1000);  % autocorrelation
subplot(2,1 ,1 ); plot(t1,x);xlabel('TIME');ylabel('X')
subplot(2,1 ,2); plot(t,R);xlabel('LAG');ylabel('Rx')
```

Resulting in:

A delta function at the origin! The magnitude should be N*10^2/(N+1) or approximately 99.9.
The next example “filters” the data sequence with a 51 tap rectangular window. The autocorrelation of the filtered waveform is plotted.

```matlab
%%
% corxmp2.m example 2 of autocorrelation calculation
% p. 225 Cooper and McGillam

clear;
close all;

rand('seed',1000);
x1=10*randn(1,1001);

h=(1/51)*ones(1,51);

x2=conv(x1,h); %length of vector is 1001+51-1
x=x2(25:25+1000); %keep vector length at 1001

t1=0:.001:1; %sampling index
[t,R]=corb(x,x,1000); %autocorrelation

subplot(2,1,1);plot(t1,x);xlabel('TIME');ylabel('X')
subplot(2,1,2); plot(t,R);xlabel('LAG');ylabel('Rx')
```

%corxmp3.m calc of standard deviation of correlation estimate

\begin{verbatim}
M = length(R);
V = (2/M)*sum(R.^2);
S = sqrt(V)
\end{verbatim}

The standard deviation of the autocorrelation just performed is:

\[ S = 0.4482 \]

From run to run, this number varies widely! Short term 0.2237 to 0.6259.

corspec.m – example code on p. 298

I have modified the supplied code to include more visualization ….

(a) The raw autocorrelation values are plotted for the original test signal and the DC removed signal.

(b) The windowed autocorrelation is shown

(c) PSD of windowed and non-windowed DC removed signal

(d) PSD of windowed and non-windowed with DC signal

(e) PSD of windowed DC and DC removed signal

If you saw the PSD related code in Appendix G here it is …. 

perspec.m – example code from appendix