Course Topics:

1. Probability
2. Random variables
3. Multiple random variables
4. Random processes
5. Linear systems

Course Objectives:

This course seeks to develop a mathematical understanding of basic statistical tools and processes as applied to inferential statistics and statistical signal processing.

1. convert an English problem description into a precise mathematical probabilistic statement (a)
2. use the general properties of random variables to solve a probabilistic problem (a, e)
3. be able to use a set of standard probability distribution functions suitable for engineering applications (a)
4. be able to calculate standard statistics from mass, distribution and density functions (a)
5. calculate confidence intervals for a population mean (a)
6. recognize and interpret a variety of deterministic and nondeterministic random processes that occur in engineering (a, b, e)
7. calculate the autocorrelation and spectral density of an arbitrary random process (a)
8. understand stochastic phenomena such as white, pink and black noise (a)
9. relate the correlation of and between input and output of autocorrelation and spectral density (a)
10. understand the mathematical characteristics of standard frequency isolation filters (a, e)
11. understand the signal-to-noise optimization principle as applied to filter design (a, e, k)
12. design Weiner and matched noise filters (a, c, e)
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1. Introduction to Probability

1.1. Engineering Applications of Probability

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1.3. Definitions of Probability
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   - Possible Outcomes
   - Trials
   - Event
   - Equally Likely Events/Outcomes
   - Objects
   - Attribute
   - Sample Space
   - With Replacement and Without Replacement

1.4. The Relative-Frequency Approach

\[ r(A) = \frac{N_A}{N} \]

\[ \Pr(A) = \lim_{N \to \infty} r(A) \]

Where \( \Pr(A) \) is defined as the probability of event A.

1.  \( 0 \leq \Pr(A) \leq 1 \)
2. \( \Pr(A) + \Pr(B) + \Pr(C) + \cdots = 1 \), for mutually exclusive events
3. An impossible event, \( A \), can be represented as \( \Pr(A) = 0 \).
4. A certain event, \( A \), can be represented as \( \Pr(A) = 1 \).

1.5. Elementary Set Theory

- Set
- Subset
- Space
- Null Set or Empty Set
- Venn Diagram
- Equality
- Sum or Union
- Products or Intersection
- Mutually Exclusive or Disjoint Sets
- Complement
- Differences
- Proofs of Set Algebra

1.6. The Axiomatic Approach
1.7. Conditional Probability

\[ \Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B), \text{ for } \Pr(B) > 0 \]

\[ \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \text{ for } \Pr(B) > 0 \]

Joint Probability

\[ \Pr(A, B) \neq \Pr(A) \] when A follows B

\[ \Pr(A, B) = \Pr(B, A) = \Pr(A \mid B) \cdot \Pr(B) = \Pr(B \mid A) \cdot \Pr(A) \]

Marginal Probabilities

Total Probability

\[ \Pr(B) = \Pr(B \mid A_1) \cdot \Pr(A_1) + \Pr(B \mid A_2) \cdot \Pr(A_2) + \cdots + \Pr(B \mid A_n) \cdot \Pr(A_n) \]

Bayes Theorem

\[ \Pr(A_i \mid B) = \frac{\Pr(B \mid A_i) \cdot \Pr(A_i)}{\Pr(B \mid A_1) \cdot \Pr(A_1) + \Pr(B \mid A_2) \cdot \Pr(A_2) + \cdots + \Pr(B \mid A_n) \cdot \Pr(A_n)} \]

1.8. Independence

\[ \Pr(A, B) = \Pr(B, A) = \Pr(A) \cdot \Pr(B) \]

1.9. Combined Experiments

1.10. Bernoulli Trials

\[ \Pr(A \text{ occurring } k \text{ times in } n \text{ trials}) = p_n(k) = \binom{n}{k} p^k \cdot q^{n-k} \]

1.11. Applications of Bernoulli Trials
2. Random Variables

2.1. Concept of a Random Variable

2.2. Distribution Functions
Probability Distribution Function (PDF)
- $0 \leq F_X(x) \leq 1$, for $-\infty < x < \infty$
- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$
- $F_X$ is non-decreasing as $x$ increases
- $Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

For discrete events
For continuous events

2.3. Density Functions
Probability Density Function (pdf)
- $f_X(x) \geq 0$, for $-\infty < x < \infty$
- $\int_{-\infty}^{x} f_X(u) \, du = 1$
- $F_X = \int_{-\infty}^{x} f_X(u) \, du$
- $Pr(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x) \, dx$

Probability Mass Function (pmf)
- $f_X(x) \geq 0$, for $-\infty < x < \infty$
- $\sum_{u=-\infty}^{\infty} f_X(u) = 1$
- $F_X(x) = \sum_{u=-\infty}^{x} f_X(u)$
- $Pr(x_1 < X \leq x_2) = \sum_{u=x_1}^{x_2} f_X(u)$

Functions of random variables
$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$
2.4. Mean Values and Moments

1st, general, nth Moments

$$
\overline{X} = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \quad \text{or} \quad E[X] = \sum_{x=-\infty}^{\infty} x \cdot \Pr(X = x)
$$

$$
E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \cdot dx \quad \text{or} \quad E[g(X)] = \sum_{x=-\infty}^{\infty} g(x) \cdot \Pr(X = x)
$$

$$
\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n \cdot f_X(x) \cdot dx \quad \text{or} \quad \overline{X^n} = E[X^n] = \sum_{x=-\infty}^{\infty} x^n \cdot \Pr(X = x)
$$

Central Moments

$$
\overline{(X - \overline{X})^m} = E[(X - \overline{X})^m] = \int_{-\infty}^{\infty} (x - \overline{X})^m \cdot f_X(x) \cdot dx
$$

$$
\overline{(X - \overline{X})^p} = E[(X - \overline{X})^p] = \sum_{x=-\infty}^{\infty} (x - \overline{X})^p \cdot \Pr(X = x)
$$

Variance and Standard Deviation

$$
\sigma^2 = \overline{(X - \overline{X})^2} = E[(X - \overline{X})^2] = \int_{-\infty}^{\infty} (x - \overline{X})^2 \cdot f_X(x) \cdot dx
$$

$$
\sigma^2 = \overline{(X - \overline{X})^2} = E[(X - \overline{X})^2] = \sum_{x=-\infty}^{\infty} (x - \overline{X})^2 \cdot \Pr(X = x)
$$

2.5. The Gaussian Random Variable

$$
f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left( -\frac{(x - \overline{X})^2}{2 \cdot \sigma^2} \right), \text{ for } -\infty < x < \infty
$$

where $\overline{X}$ is the mean and $\sigma$ is the variance

$$
F_X(x) = \int_{y=-\infty}^{x} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left( -\frac{(v - \overline{X})^2}{2 \cdot \sigma^2} \right) \cdot dv
$$

Unit Normal (Appendix D)

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{u=-\infty}^{x} \exp\left( -\frac{u^2}{2} \right) \cdot du
$$

$$
\Phi(-x) = 1 - \Phi(x)
$$

$$
F_X(x) = \Phi\left( \frac{x - \overline{X}}{\sigma} \right) \quad \text{or} \quad F_X(-x) = 1 - \Phi\left( \frac{x - \overline{X}}{\sigma} \right)
$$
The *Q-function* is the complement of the normal function, \( \Phi \): (Appendix E)

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{u^2}{2}\right) \, du
\]

2.6. Density Functions Related to Gaussian

2.7. Other Probability Density Functions

**Exponential Distribution**

\[
f_T(\tau) = \begin{cases} 
\frac{1}{M} \cdot \exp\left(-\frac{\tau}{M}\right), & \text{for } 0 \leq \tau \\
0, & \text{for } \tau < 0
\end{cases}
\]

\[
F_T(\tau) = 1 - \exp\left(-\frac{\tau}{M}\right), \quad \text{for } 0 \leq \tau
\]

\[
\bar{T} = E[T] = M
\]

\[
\bar{T}^2 = E[T^2] = 2 \cdot M^2
\]

\[
E\left[(\tau - \bar{T})^2\right] = \sigma_T^2 = \bar{T}^2 - E[T]^2 = 2 \cdot M^2 - (M)^2 = M^2
\]

**Binomial Distribution**

\[
f_B(x) = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k} \cdot \delta(x - k)
\]

\[
F_B(x) = \sum_{k=0}^{\min(x,n)} \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k} \cdot u(x - k)
\]

2.8. Conditional Probability Distribution and Density Functions

\[
\Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B), \quad \text{for } \Pr(B) > 0
\]

\[
\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \text{for } \Pr(B) > 0
\]

\[
\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A,B)}{\Pr(B)}, \quad \text{for } \Pr(B) > 0
\]

It can be shown that \( F(x \mid M) \) is a valid probability distribution function with all the expected characteristics:

- \( 0 \leq F(x \mid M) \leq 1 \), for \(-\infty < x < \infty\)
- \( F(-\infty \mid M) = 0 \) and \( F(\infty \mid M) = 1 \)
- \( F(x \mid M) \) is non-decreasing as \( x \) increases
- \( \Pr(x_1 < X \leq x_2 \mid M) = F(x_2 \mid M) - F(x_1 \mid M) \)
2.9. Examples and Applications

3. Several Random Variables

3.1. Two Random Variables

Joint Probability Distribution Function (PDF)

\[ F(x, y) = \Pr(X \leq x, Y \leq y) \]

- \( 0 \leq F(x, y) \leq 1 \), \( \text{for } -\infty < x < \infty \text{ and } -\infty < y < \infty \)
- \( F(-\infty, y) = F(x, -\infty) = F(-\infty, -\infty) = 0 \)
- \( F(\infty, \infty) = 1 \)
- \( F(x, y) \) is non-decreasing as either \( x \) or \( y \) increases
- \( F(x, \infty) = F_X(x) \) and \( F(\infty, y) = F_Y(y) \)

Joint Probability Density Function (pdf)

\[ f(x, y) = \frac{\partial^2 F_X(x)}{\partial x \partial y} \]

- \( f(x, y) \geq 0 \), \( \text{for } -\infty < x < \infty \text{ and } -\infty < y < \infty \)
- \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot dx \cdot dy = 1 \)
- \( F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) \cdot du \cdot dv \)
- \( f_X(x) = \int_{-\infty}^{\infty} f(x, y) \cdot dy \) and \( f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \cdot dx \)
- \( \Pr(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \cdot dx \cdot dy \)

Expected Values

\[ E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y) \cdot dx \cdot dy \]

Correlation

\[ E[X \cdot Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f(x, y) \cdot dx \cdot dy \]
3.2. Conditional Probability--Revisited

\[ F_X(x \mid Y \leq y) = \frac{Pr(X \leq x \mid M)}{Pr(M)} = \frac{F(x, y)}{F_Y(y)} \]

\[ F_X(x \mid y_1 \leq Y \leq y_2) = \frac{F(x, y_2) - F(x, y_1)}{F_Y(y_2) - F_Y(y_1)} \]

\[ F_X(x \mid Y = y) = \frac{f(x, y)}{f_Y(y)} \]

\[ F_Y(y \mid X = x) = \frac{f(x, y)}{f_X(x)} \]

\[ f(x \mid y) = \frac{f(y \mid x) \cdot f_X(x)}{f_Y(y)} \]

\[ f(x, y) = f(x \mid Y = y) \cdot f_Y(y) = f(y \mid X = x) \cdot f_X(x) \]

\[ f(x \mid Y = y) = \frac{f(y \mid X = x) \cdot f_X(x)}{f_Y(y)} \]

\[ f(y \mid X = x) = \frac{f(x \mid Y = y) \cdot f_Y(y)}{f_X(x)} \]

3.3. Statistical Independence

\[ f(x, y) = f_X(x) \cdot f_Y(y) \]

\[ E[X \cdot Y] = E[X] \cdot E[Y] = \overline{X} \cdot \overline{Y} \]

3.4. Correlation between Random Variables

\[ E[X \cdot Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f(x, y) \cdot dx \cdot dy \]

Covariance

\[ E[(X - E[X]) \cdot (Y - E[Y])] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X) \cdot (y - \mu_Y) \cdot f(x, y) \cdot dx \cdot dy \]

Correlation coefficient or normalized covariance,

\[ \rho = E \left[ \left( \frac{X - \mu_X}{\sigma_X} \right) \cdot \left( \frac{Y - \mu_Y}{\sigma_Y} \right) \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{x - \mu_X}{\sigma_X} \right) \cdot \left( \frac{y - \mu_Y}{\sigma_Y} \right) \cdot f(x, y) \cdot dx \cdot dy \]

\[ \rho = \frac{E[x \cdot y] - \mu_X \cdot \mu_Y}{\sigma_X \cdot \sigma_Y} \]
3.5. Density Function of the Sum of Two Random Variables

\[ Z = X + Y \]

\[ F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) \cdot du \cdot dv \]

\[ F_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \cdot \int_{-\infty}^{z-y} f_X(x) \cdot dx \cdot dy \]

\[ f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x) \cdot dx = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y) \cdot dy \]

3.6. Probability Density Function of a Function of Two Random Variables

3.7. The Characteristic Function

\[ \phi(u) = E[\exp(j \cdot u \cdot X)] \]

\[ \phi(u) = \int_{-\infty}^{\infty} f(x) \cdot \exp(j \cdot u \cdot x) \cdot dx \]

The inverse of the characteristic function is then defined as:

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u) \cdot \exp(-j \cdot u \cdot x) \cdot du \]

Computing other moments is performed similarly, where:

\[ \frac{d^n[\phi(u)]}{du^n} = \int_{-\infty}^{\infty} f(x) \cdot (j \cdot x)^n \cdot \exp(j \cdot u \cdot x) \cdot dx \]

\[ \frac{d^n[\phi(u)]}{du^n} \bigg|_{u=0} = \int_{-\infty}^{\infty} (j \cdot x)^n \cdot f(x) \cdot dx = j^n \cdot \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx = j^n \cdot E[X^n] \]
4. Elements of Statistics

4.1. Introduction

4.2. Sampling Theory--The Sample Mean

Sample Mean \( \hat{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \), where \( X_i \) are random variables with a pdf.

Variance of the sample mean
\[
Var(\hat{X}) = \frac{1}{n} \left( \frac{1}{n} \sum_{i=1}^{n} (X_i^2) + \frac{-n}{n^2} \sum_{i=1}^{n} (X_i)^2 \right) = \frac{(X_i^2) - (X_i)^2}{n} = \frac{\sigma^2}{n} \\
Var(\hat{X}) = \frac{\sigma^2}{n} \cdot \left( \frac{N-n}{N-1} \right)
\]

4.3. Sampling Theory--The Sample Variance

\( S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{X})^2 \)

\[
E[S^2] = \frac{n-1}{n} \cdot \sigma^2 \\
E[S^2] = \frac{N}{N-1} \cdot \frac{n-1}{n} \cdot \sigma^2
\]

unbiased
\[
E[\hat{S}^2] = \frac{n}{n-1} \cdot E[S^2] = \frac{n}{n-1} \cdot \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{X})^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{X})^2 \\
E[\hat{S}^2] = \frac{N}{N-1} \cdot \frac{n}{n-1} \cdot E[S^2]
\]

4.4. Sampling Distributions and Confidence Intervals

Gaussian
\[
Z = \frac{\hat{X} - \bar{X}}{\sigma / \sqrt{n}}
\]

Student’s t distribution
\[
T = \frac{\hat{X} - \bar{X}}{\hat{S} / \sqrt{n-1}} = \frac{\hat{X} - \bar{X}}{\hat{S} / \sqrt{n}} \\
\bar{X} - \frac{k \cdot \sigma}{\sqrt{n}} \leq \hat{X} \leq \bar{X} + \frac{k \cdot \sigma}{\sqrt{n}}
\]
4.5. Hypothesis Testing
One tail or two-tail testing

4.6. Curve Fitting and Linear Regression

\[ a = \frac{\hat{y} \cdot R_{XX} \cdot \hat{x} - \hat{y} \cdot R_{XY} \cdot \hat{x}}{R_{XX} - \hat{x}^2} \]

\[ b = \frac{R_{XY} \cdot (\hat{y} \cdot \hat{x}) - C_{XX}}{R_{XX} - \hat{x}^2} \]

4.7. Correlation between Two Sets of Data

\[ \mu_X = E[X] = \frac{1}{n} \sum_{i=1}^{n} x_i \]

\[ E[X^2] = R_{XX} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \]

\[ \sigma_X^2 = C_{XX} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 = R_{XX} - \mu_X^2 \]

\[ R_{XY} = E[X \cdot Y] = \frac{1}{n} \sum_{i=1}^{n} x_i \cdot y_i \]

\[ C_{XY} = E[(X - \bar{X}) \cdot (Y - \bar{Y})] = \frac{1}{n} \sum_{i=1}^{n} (x_i \cdot y_i) - \bar{X} \cdot \bar{Y} \]

\[ \rho_{XY} = \frac{C_{XY}}{\sigma_X \cdot \sigma_Y} \]
5. Random Processes

5.1. Introduction

Ensemble

For example, assume that there is a known AM signal transmitted:
\[ s(t) = (1 + b \cdot A(t)) \cdot \sin(\omega \cdot t) \]
at an undetermined distance the signal is received as
\[ y(t) = (1 + b \cdot A(t)) \cdot \sin(\omega \cdot t + \theta), \quad 0 \leq \theta \leq 2 \cdot \pi \]
The received signal is mixed and low pass filtered …
\[ x(t) = h(t) \otimes [y(t) \cdot \cos(\omega \cdot t)] = h(t) \otimes [(1 + b \cdot A(t)) \cdot \sin(\omega \cdot t + \theta) \cdot \cos(\omega \cdot t)], 0 \leq \theta \leq 2 \cdot \pi \]
If the filter removes the 2wt term, we have
\[ x(t) = h(t) \otimes [y(t) \cdot \cos(\omega \cdot t)] = \frac{(1 + b \cdot A(t))}{2} \cdot \sin(\theta), 0 \leq \theta \leq 2 \cdot \pi \]
Notice that based on the value of the random variable, the output can change significantly! From producing no output signal, \((\theta = 0, \pi)\), to having the output be positive or negative \((\theta = 0 \text{ to } \pi \text{ or } \pi \text{ to } 2 \pi)\). P.S. This is not how you perform non-coherent AM demodulation.

To perform coherent AM demodulation, all I need to do is measured the value of the random variable and use it to insure that the output is a maximum (i.e. mix with \(\cos(\omega \cdot t + \theta_m)\), where \(\theta_m = \theta(t_1)\)).

5.2. Continuous and Discrete Random Processes

5.3. Deterministic and Nondeterministic Random Processes

5.4. Stationary and Nonstationary Random Processes

The requirement that all marginal and joint density functions be independent of the choice of time origin is frequently more stringent (tighter) than is necessary for system analysis. A more relaxed requirement is called stationary in the wide sense: where the mean value of any random variable is independent of the choice of time, \(t\), and that the correlation of two random variables depends only upon the time difference between them. That is
\[ E[X(t)] = \overline{X} = \mu_X \quad \text{and} \]
\[ E[X(t_1) \cdot X(t_2)] = E[X(0) \cdot X(t_2 - t_1)] = \overline{X(0)} \cdot \overline{X(\tau)} = R_{XX}(\tau) \text{ for } \tau = t_2 - t_1 \]
You will typically deal with Wide-Sense Stationary Signals.
5.5. Ergodic and Nonergodic Random Processes
A Process for Determining Stationarity and Ergodicity

a) Find the mean and the 2nd moment based on the probability

b) Find the time sample mean and time sample 2nd moment based on time averaging.

c) If the means or 2nd moments are functions of time … non-stationary

d) If the time average mean and moments are not equal to the probabilistic mean and moments \textbf{or} if it is not stationary, then it is non-ergodic.

For ergodic processes, all the statistics can be determined from a single function of the process. This may also be stated based on the time averages. For an ergodic process, the time averages (expected values) equal the ensemble averages (expected values). That is to say,

\[
\overline{X^n} = \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t) \cdot dt
\]

Note that ergodicity cannot exist unless the process is stationary!

5.6. Measurement of Process Parameters

5.7. Smoothing Data with a Moving Window Average

A Process for Determining Stationarity and Ergodicity

a) Find the mean and the 2nd moment based on the probability

b) Find the time sample mean and time sample 2nd moment based on time averaging.

c) If the means or 2nd moments are functions of time … non-stationary

d) If the time average mean and moments are not equal to the probabilistic mean and moments \textbf{or} if it is not stationary, then it is non-ergodic.
6. Correlation Functions

6.1. Introduction

6.2. Example: Autocorrelation Function of a Binary Process

\[ R_{XX}(t_1, t_2) = E[X_1X_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdot \{x_1x_2f(x_1, x_2)\} \]

The above function is valid for all processes, stationary and non-stationary. For WSS processes:

\[ R_{XX}(t_1, t_2) = E[X(t)X(t+\tau)] = R_{XX}(\tau) \]

If the process is ergodic, the time average is equivalent to the probabilistic expectation, or

\[ \mathbb{R}_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t + \tau) \cdot dt = \langle x(t) \cdot x(t + \tau) \rangle \]

and

\[ \mathbb{R}_{XX}(\tau) = R_{XX}(\tau) \]

6.3. Properties of Autocorrelation Functions

1) \[ R_{xx}(0) = E[X^2] = \overline{X^2} \quad \text{or} \quad \mathbb{R}_{xx}(0) = \langle x(t)^2 \rangle \]
2) \[ R_{XX}(\tau) = R_{XX}(-\tau) \]
3) \[ |R_{XX}(\tau)| \leq R_{XX}(0) \]
4) If X has a DC component, then Rxx has a constant factor.
5) If X has a periodic component, then Rxx has a will also have a periodic component of the same period.
6) If X is ergodic and zero mean and has no periodic component, then

\[ \lim_{|\tau| \to \infty} R_{XX}(\tau) = 0 \]

7) Autocorrelation functions can not have an arbitrary shape. One way of specifying shapes permissible is in terms of the Fourier transform of the autocorrelation function. That is, if

\[ \mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-j\omega t) \cdot dt \]

then the restriction states that

\[ \mathcal{F}[R_{XX}(\tau)] \geq 0 \quad \text{for all } \omega \]

6.4. Measurement of Autocorrelation Functions

6.5. Examples of Autocorrelation Functions

6.6. Crosscorrelation Functions

The cross-correlation is defined as:

\[
R_{XY}(t_1, t_2) = \mathbb{E}[X_1 Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_2 \cdot \{x_1 y_2 f(x_1, y_2)\}
\]

\[
R_{YX}(t_1, t_2) = \mathbb{E}[Y_1 X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \cdot \{y_1 x_2 f(y_1, x_2)\}
\]

For jointly WSS processes:

\[
R_{XY}(t_1, t_2) = \mathbb{E}[X(t)Y(t + \tau)] = R_{XY}(\tau)
\]

\[
R_{YX}(t_1, t_2) = \mathbb{E}[Y(t)X(t + \tau)] = R_{YX}(\tau)
\]

and

\[
\Re_{XY}(\tau) = R_{XY}(\tau)
\]

\[
\Re_{YX}(\tau) = R_{YX}(\tau)
\]

6.7. Properties of Cross-correlation Functions

1) The properties of the zeroth lag have no particular significance and do not represent mean-square values. It is true that the ordered crosscorrelations are equal at 0.

\[
R_{XY}(0) = R_{YX}(0) \quad \text{or} \quad \Re_{XY}(0) = \Re_{YX}(0)
\]

2) Crosscorrelation functions are not generally even functions. There is an antisymmetry to the ordered crosscorrelations:

\[
R_{XY}(\tau) = R_{YX}(-\tau)
\]

3) The crosscorrelation does not necessarily have its maximum at the zeroth lag. This makes sense if you are correlating a signal with a timed delayed version of itself. The crosscorrelation should be a maximum when the lag equals the time delay!

4) If X and Y are statistically independent, then the ordering is not important

\[
R_{XY}(\tau) = \mathbb{E}[X(t) \cdot Y(t + \tau)] = \mathbb{E}[X(t)] \cdot \mathbb{E}[Y(t + \tau)] = \overline{X} \cdot \overline{Y}
\]

and

\[
R_{XY}(\tau) = \overline{X} \cdot \overline{Y} = R_{YX}(\tau)
\]

5) If X is a stationary random process and it’s differentiable with respect to time, the crosscorrelation of the signal and it’s derivative is given by

\[
R_{X\dot{X}}(\tau) = \frac{d R_{XX}(\tau)}{d\tau}
\]

6.8. Examples and Applications of Crosscorrelation Functions

6.9. Correlation Matrices for Sampled Functions
7. Spectral Density

7.1. Introduction

Therefore, we can define a power spectral density for the ensemble as:

\[
S_{XX}(w) = \mathbb{E}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot \exp(-iw\tau) \cdot d\tau
\]

\[
S_{XX}(w) = \mathbb{E}[R_{XX}(\tau)] = R_{XX}(t) = \mathbb{E}^{-1}[S_{XX}(w)]
\]

\[
R_{XX}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(iwt) \cdot dw
\]

7.2. Relation of Spectral Density to the Fourier Transform

\[
\mathbb{E}[\Re R_{XX}(\tau)] = \mathbb{E}[X(\tau)] \cdot \Re X(-w) = |\Re X(w)|^2
\]

7.3. Properties of Spectral Density

The power spectral density as a function is always

- real,
- positive,
- and an even function in \( w \).

7.4. Spectral Density and the Complex Frequency Plane

7.5. Mean-Square Values From Spectral Density

The mean squared value of a random process is equal to the 0\textsuperscript{th} lag of the autocorrelation

\[
E[X^2] = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot \exp(iw \cdot 0) \cdot dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) \cdot dw
\]

\[
E[X^2] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f) \cdot \exp(i2\pi f \cdot 0) \cdot df = \int_{-\infty}^{\infty} S_{XX}(f) \cdot df
\]

As a note, since the PSD is real and symmetric, the integral can be performed as
\[ E[X^2] = R_{XX}(0) = 2\cdot\frac{1}{2\pi}\int_0^\infty S_{XX}(w)\cdot dw \]
\[ E[X^2] = R_{XX}(0) = 2\int_0^\infty S_{XX}(f)\cdot df \]

7.6. Relation of Spectral Density to the Autocorrelation Function

The Fourier Transform in \( f \)
\[ S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cdot \exp(-i2\pi f\tau)\cdot d\tau \]
\[ R_{XX}(t) = \int_{-\infty}^{\infty} S_{XX}(f)\cdot \exp(i2\pi ft)\cdot df \]

7.7. White Noise

As a result, we define “White Noise” as
\[ R_{XX}(\tau) = S_0 \cdot \delta(t) \]
\[ S_{XX}(w) = S_0 = \frac{N_0}{2} \]

7.8. Cross-Spectral Density

The Fourier Transform in \( w \)
\[ S_{XY}(w) = \int_{-\infty}^{\infty} R_{XY}(\tau)\cdot \exp(-iw\tau)\cdot d\tau \quad \text{and} \quad S_{YX}(w) = \int_{-\infty}^{\infty} R_{YX}(\tau)\cdot \exp(-iwt)\cdot d\tau \]
\[ R_{XY}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(w)\cdot \exp(iwt)\cdot dw \quad \text{and} \quad R_{YX}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(w)\cdot \exp(iwt)\cdot dw \]

Properties of the functions
\[ S_{XY}(w) = \text{conj}(S_{YX}(w)) \]
Since the cross-correlation is real,
- the real portion of the spectrum is even
- the imaginary portion of the spectrum is odd
7.9. Autocorrelation Function Estimate of Spectral Density

7.10. Periodogram Estimate of Spectral Density

7.11. Examples and Applications of Spectral Density
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