Chapter 9: Random Signals and Noise

- Random processes
- Random signals
- Noise
- Baseband signal transmission with noise
- Baseband pulse transmission with noise
Ensemble

Waveforms in an ensemble $v(t,s)$

Figure 9.1-1
Example 9.1-1 Random Phase

- The phase at which a cosine wave is received is assumed to be a uniformly distributed random variable.

\[ v(t) = \cos(2\pi \cdot f_c \cdot t + \Theta) \quad p_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta < 2\pi \]

\[ E[v(t)] = \int_{-\infty}^{\infty} \cos(2\pi \cdot f_c \cdot t + \theta) \cdot p_\Theta(\theta) \cdot d\theta \]

\[ E[v(t)] = \frac{1}{2\pi} \cdot \int_{0}^{2\pi} \cos(2\pi \cdot f_c \cdot t + \theta) \cdot d\theta = \frac{1}{2\pi} \cdot \frac{1}{2} \cdot \left[ e^{-j2\pi f_c t - j\theta} + e^{j2\pi f_c t + j\theta} \right] \cdot d\theta \]

\[ E[v(t)] = \frac{1}{2\pi} \cdot \frac{1}{2} \left[ e^{-j2\pi f_c t} \cdot \frac{e^{-j\theta}}{-j} + e^{j2\pi f_c t} \cdot \frac{e^{j\theta}}{+j} \right]_{0}^{2\pi} \]

\[ E[v(t)] = \frac{1}{2\pi} \cdot \frac{1}{2j} \cdot \left[ -e^{-j2\pi f_c t} \cdot (1-1) + e^{j2\pi f_c t} \cdot (1-1) \right] = 0 \]
Random Digital Signal

- A random digital waveform based on a rectangular pulse train with bit/symbol period $D$
  - The time delay is a continuous RV uniformly distributed between 0 and the pulse width $D$
  - The amplitude of the $k$th pulse is a discrete RV with zero mean and a known variance.
  - The amplitudes during different intervals are independent (i.i.d)

$$v(t) = \sum_{k=-\infty}^{\infty} a_k \cdot \text{rect}\left(\left\{ \frac{t - T_d + \frac{D}{2} - k \cdot D}{D} \right\} \right)$$

$$p(T_d) = \frac{1}{D}, \quad 0 < T_d \leq D$$

$$E[a_n] = 0, E[a_n^2] = \sigma^2$$

$$E[a_j \cdot a_k] = 0, \quad \text{for } j \neq k$$
Random Digital Signal (2)

\( v_i(t) \)

\( a_0 \)

\( 0 \quad T_d \quad a_1 \quad a_2 \quad D \)

\( t \)

\[ v(t) = \sum_{k=-\infty}^{\infty} a_k \cdot \text{rect} \left( \frac{t - T_d + D/2 - k \cdot D}{D} \right) \]

\( t_1 \quad kD + T_d \quad t_2 \quad (k + 1)D + T_d \)

(b)
Random Digital Signal

- **Autocorrelation**

\[ R_{vv}(\tau) = E[v(t) \cdot v(t + \tau)] = \sigma^2 \cdot \left( 1 - \frac{|\tau|}{D} \right), \quad -D < \tau < D \]

- **Power Spectral Density**

\[ S_{vv}(w) = \Im \{E[v(t) \cdot v(t + \tau)]\} = \sigma^2 \cdot D \cdot \sin^2 (f \cdot D) \]
Random Telegraph: Example 9.2-1

(a) Sample function; (b) Autocorrelation; (c) Power spectrum:

Figure 9.2-3

- Poisson distribution
  - Average number of transitions per unit time
  - DC component
Random Telegraph: Example 9.2-1

- Poisson distribution

\[ R_{vv}(\tau) = \frac{A^2}{4} (1 + \exp(-2 \cdot \mu \cdot |t|)) \]

\[ \sigma_{vv}^2 = E[v(t)^2] - E[v(t)]^2 \]

\[ \sigma_{vv}^2 = \frac{A^2}{4} (1 + \exp(-2 \cdot \mu \cdot 0)) - \frac{A^2}{4} (1 + \exp(-2 \cdot \mu \cdot \infty)) \]

\[ \sigma_{vv}^2 = 2 \cdot \frac{A^2}{4} - \frac{A^2}{4} = \frac{A^2}{4} \]

\[ S_{vv}(f) = \frac{A^2}{4 \cdot \mu \cdot \left[ 1 + \left( \frac{\pi \cdot f}{\mu} \right)^2 \right]} + \frac{A^2}{4} \cdot \delta(f) \]
Modulation

- Typical receivers have a random phase offset in the carrier. Uniform distribution.

\[ z(t) = v(t) \cdot \cos(2\pi f_c \cdot t + \Phi) \]

\[ p(\Phi) = \frac{1}{2\pi}, \quad 0 \leq \Phi < 2\pi \]

\[ R_{zz}(\tau) = \frac{1}{2} \cdot R_{vv}(\tau) \cdot \cos(2\pi f_c \cdot \tau) \]

\[ S_{zz}(f) = \frac{1}{4} \cdot \left[ S_{vv}(f - f_c) + S_{vv}(f + f_c) \right] \]
Linear Filtering

• Filtering of random variables

\[ R_{yx}(\tau) = h(\tau) * R_{xx}(\tau) \]

\[ R_{yy}(\tau) = h(-\tau) * h(\tau) * R_{xx}(\tau) \]

\[ S_{yy}(f) = H(f)^* \cdot H(f) \cdot S_{xx}(f) \]

\[ S_{yy}(f) = |H(f)|^2 \cdot S_{xx}(f) \]
Thermal Noise Power

• Noise produced by the random motion of charge particles in conducting media
• Modeled as additive white Gaussian noise (AWGN)

\[ N = \kappa \cdot T \cdot B \]

– Where \( \kappa \) is Boltzmann’s constant
– \( T \) is absolute temperature in degrees Kelvin
– \( B \) is the bandwidth in Hertz

\[ \kappa = -228.6 \text{ dBW / K \cdot Hz} \]

\[ T_0 = 290 \text{°K IEEE ref} \]

\[ N_0 = \kappa \cdot T_0 = 1.38e-23 \cdot 290 \approx 4.00e-21 \]

\[ N_0 = -204 \text{ dBW / Hz} = -174 \text{ dBm / Hz} \]
Thermal Resistance Noise

Resistor Models with Noise PSD

(a) Thevenin Voltage Model

\[ G_v(f) = \frac{\frac{1}{2RkT}}{2RkT} \]

(b) Norton Current Model

\[ G_i(f) = \frac{\frac{1}{2kT}}{R} \]

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Noise Approximation

- **Uniform Noise Spectral Density**
  - Resistor description (Thevenin Model)
    \[ G_{vv}(f) = 2 \cdot R \cdot \kappa \cdot T \]

- **Available Power from the “noise source”**
  - Source output power into a matched load
    \[
    V_{sout} = \frac{R}{2 \cdot R} V_s \\
    P_{sout} = \frac{(V_{sout})^2}{R} = \left(\frac{V_s}{2}\right)^2 \cdot \frac{1}{R} = \frac{V_s^2}{4 \cdot R}
    \]

\[
G_{ss}(f) = \frac{G_{vv}(f)}{4 \cdot R} = \frac{2 \cdot R \cdot \kappa \cdot T}{4 \cdot R} = \frac{\kappa \cdot T}{2} = \frac{N_0}{2}
\]

\[
R_{ss}(\tau) = \frac{N_0}{2} \cdot \delta(\tau)
\]
System Noise

- Since the noise power spectrum is uniform, a systems noise power is the product of the noise power and the integral of the filter power.

\[
S_{NN}(f) = |H(f)|^2 \cdot S_{N_0N_0}(f) = \frac{N_0}{2} \cdot |H(f)|^2
\]

\[
R_{NN}(0) = \frac{N_0}{2} \cdot \int_{-\infty}^{\infty} |H(f)|^2 \cdot df = N_0 \cdot \int_{0}^{\infty} |H(f)|^2 \cdot df
\]
Noise Equivalent Bandwidth

• If we want the total noise power after the filter, we can integrate the PSD for all frequencies or use the Filtered noise autocorrelation function at zero.
  – Both of these approaches may be difficult
  – Could we generate a more simple “noise equivalent bandwidth for filters” that is rectangular?
Noise Equivalent Bandwidth

\[
R_{NN}(0) = \frac{N_0}{2} \cdot \int_{-\infty}^{\infty} |H(f)|^2 \cdot df = \frac{N_0}{2} \cdot \int_{0}^{\infty} |H(f)|^2 \cdot df
\]

- When filtering, it is convenient to think of band-limited noise, where the filter is a rect function with bandwidth \( B_{EQN} \)

\[
H_{rect\_model}(f) = \sqrt{\text{Gain}_{DC\_Power}} \cdot \text{rect} \left( \frac{f}{2 \cdot B_{EQN}} \right) \quad \text{Gain}_{DC\_Power} = |H(0)|^2
\]

\[
\int_{0}^{\infty} |H(f)|^2 \cdot df \approx \int_{0}^{\infty} |H(f)_{rect\_model}|^2 \cdot df = \text{Gain}_{DC\_Power} \cdot B_{EQN} = |H(0)|^2 \cdot B_{EQN}
\]

\[
B_{EQN} = \frac{\int_{0}^{\infty} |H(f)|^2 \cdot df}{|H(0)|^2}
\]
Noise Equivalent Bandwidth

• Low pass filter

\[
\text{coherent\_gain} = |H(0)| \quad \text{Gain}_{\text{DC\_power}} = |H(0)|^2
\]

\[
B_{\text{EQN}} = \int_{0}^{\infty} |H(f)|^2 \cdot df
\]

\[
P_N = R_{\text{NN}}(0) = \frac{N_0}{2} \cdot 2 \cdot |H(0)|^2 \cdot B_{\text{EQN}} = |H(0)|^2 \cdot N_0 \cdot B_{\text{EQN}}
\]

• For a unity gain filter

– assumed when computing receiver input noise power

\[
B_{\text{EQN}} = \int_{0}^{\infty} |H(f)|^2 \cdot df
\]

\[
P_N = R_{\text{NN}}(0) = \frac{N_0}{2} \cdot 2 \cdot B_{\text{EQN}} = N_0 \cdot B_{\text{EQN}}
\]
Model of Received Signal with Noise

\[ y_D(t) = x_D(t) + n_D(t) \]
Signal Plus Noise

Analog baseband transmission system with noise: Figure 9.4-2

- Additive Gaussian White Noise
Signal-to-Noise Ratio

- Comparing the desired signal power to the undesired noise power.

\[ y(t) = x_c(t) + n(t) \]

- To compare signal and noise power, we must assume a filtering operations.
Signal-to-Noise Ratio

\[ y_D(t) = [x_R(t) + n(t)] * h(t) \]

- Equivalent receiver input signal and noise (ER)
  \[ y_R(t) = x_{ER}(t) + n_{ER}(t) \]
- Equivalent destination signal and noise (D) or predemodulation (PreD)
  \[ y_{PreD}(t) = x_{PreD}(t) + n_{PreD}(t) \]
Signal-to-Noise Ratio

- Equivalent receiver input SNR (ER)

\[
SNR_R \equiv \frac{E[x_{ER}(t)^2]}{E[n_{ER}(t)^2]} \approx \frac{E[x_{ER}(t)^2]}{N_0 \cdot B_{EQN}} \approx \frac{S_R}{N_0 \cdot B_{EQN}}
\]

- Equivalent destination SNR

\[
SNR_R = \frac{E[x_{PreD}(t)^2]}{E[n_{PreD}(t)^2]} \approx \frac{S_D}{N_D} \approx \frac{S_D}{\eta \cdot N_0 \cdot B_{EQN}}
\]

\(\eta\) can be used to represent receiver noise figure contributions.
Increase in SNR with filtering

• If a filter matched to the input signal is applied, the noise power would be reduced to the smallest equivalent noise bandwidth that is allowed.
  – Filter to minimize noise power
  – Importance of the IF filter in a super-het receiver!

• Front-end filtering goals – a dilemma
  – Minimize signal power loss (wider bandwidth)
  – Minimize filter equivalent noise bandwidth (narrower bandwidths)
  – A trade-off must be made!
## Typical Transmission Requirements

<table>
<thead>
<tr>
<th>Signal Type</th>
<th>Freq. Range</th>
<th>SNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intelligible Voice</td>
<td>500 Hz to 2 kHz</td>
<td>5-10</td>
</tr>
<tr>
<td>Telephone Quality</td>
<td>200 Hz to 3.2 kHz</td>
<td>25-35</td>
</tr>
<tr>
<td>AM Broadcast Audio</td>
<td>100 Hz to 5 kHz</td>
<td>40-50</td>
</tr>
<tr>
<td>High-fidelity Audio</td>
<td>20 Hz to 20 kHz</td>
<td>55-65</td>
</tr>
<tr>
<td>Video</td>
<td>60 Hz to 4.2 MHz</td>
<td>45-55</td>
</tr>
<tr>
<td>Spectrum Analyzer</td>
<td>100 kHz-1.8 GHz</td>
<td>65-75</td>
</tr>
</tbody>
</table>
Pulse Measurement in Noise

\[ p(t) \rightarrow + \rightarrow \text{LPF} \quad B_N \geq \frac{1}{2\tau} \rightarrow y(t) = p(t) + n(t) \]

\[ G(f) = \frac{N_0}{2} \]

(a)

(b)
Pulse Measurement in Noise

Let received pulse be a rectangle with

\[ A = \text{amplitude}, \tau = \text{duration}, \quad E_p = A^2\tau = \text{energy}, \]

and zero mean noise with PSD \( G(f) = N_0 / 2 \)

\( \text{LPF} \Rightarrow B \approx B_N \geq 1/2\tau \)

At the filter output \( \Rightarrow y(t_a) = A + n(t_a) = A + \varepsilon_A \)

where \( \varepsilon_A \Rightarrow \text{amplitude error} \)

\[ \Rightarrow \sigma_A^2 = n^2 = N_0B_N \]

Lower bound on error variance \( \Rightarrow \sigma_A^2 \geq \frac{N_0}{2\tau} = \frac{N_0A^2}{2E_p} \)

Achieving lower bound \( \Rightarrow \text{matched filter} \)

Note: \( y \) is a random variable with a variance and pdf

\[ y = A \pm \sigma_A \]
Time-position Error
Caused By Noise

• Time error rv relation
  \[ \frac{\varepsilon_t}{n(t_b)} = \frac{t_r}{A} \]

• Time error variance
  \[ E[\varepsilon_t^2] = \sigma_t^2 = E \left[ \left( \frac{t_r}{A} \cdot n(t_b) \right)^2 \right] \]
Time-position Error
Caused By Noise (2)

- Time Position Error Variance
  \[ E[\varepsilon_t^2] = \sigma_t^2 = E \left[ \left( \frac{t_r}{A} \cdot n(t_b) \right)^2 \right] = \left( \frac{t_r}{A} \right)^2 \cdot N_0 \cdot B_N \]

  \[ \sigma_t^2 \approx \frac{N_0 \cdot B_N \cdot \tau}{4 \cdot E_p \cdot B_T^2}, \quad \text{for } B_T \neq B_N \]

- Matching the transmit bandwidth
  \[ \sigma_t^2 \approx \frac{N_0 \cdot \tau}{4 \cdot E_p \cdot B_T} = \frac{N_0}{4 \cdot A^2 \cdot B_T}, \quad \text{for } B_T = B_N \]

- Therefore
  \[ t_{edge} = t_{measure} \pm \sigma_t \]

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Matched Filter

• See Chapter 9 of ECE 3800 Text
  – Notes are on the web site