WESTERN MICHIGAN UNIVERSITY
Department of Electrical and Computer Engineering

Analog and RF Filters Design Manual:
A Filter Design Guide by and for WMU Students

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Abstract: Students, practicing engineers, hobbyists, and researchers use a wide range of circuits as fundamental building blocks. This manual provides numerous analog circuits for study and implementation, many of which have been building blocks for circuitry built and tested at WMU.

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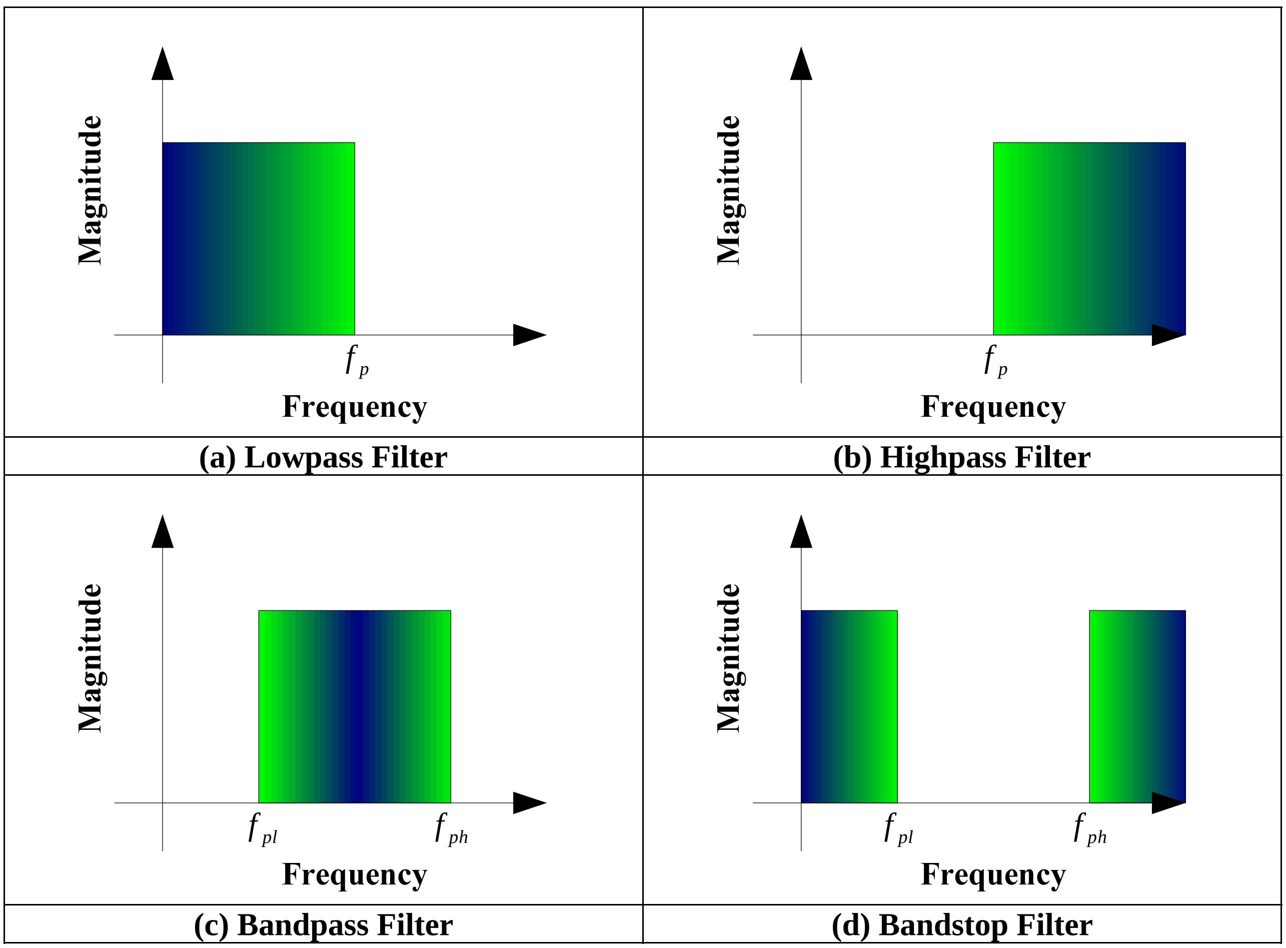
1 Introduction

Filtering, whether intended or unintended, modifies the magnitude and phase of signal frequency components. Every analog or radio frequency (RF) circuit performs filtering on the signals passing through them. Therefore for RF or analog circuit designer, it is important to understand, how to design and construct filters.

1.1 General Types of Filters

Filter types are defined based on how they modify the magnitude and/or phase of sinusoidal frequency components. In most cases, the primary concern is the magnitude response, which will be addressed in this manual. For a special class of filters concerned with phase modifications, information on "all-pass filters" can be found in a number of reference [mitra, harris, etc].

Filters are typically classified based on how they modify the frequency spectrum. The four basic types of filters are; the lowpass filter, highpass filter, bandpass filter and bandstop filter. Idealized versions of these filters are shown in Fig. 1.



(a) Lowpass Filter

(b) Highpass Filter

(c) Bandpass Filter

(d) Bandstop Filter

Figure 1: General classification of filters

The lowpass filter (LPF), figure 1a, is used to allow low frequency signals, below a cutoff frequency f_p, to pass through the filter, while attenuating or stopping higher frequencies. The lowpass filter is the most common filter, used for reducing high-frequency noise, allowing more accurate measurements of low frequency signals, and limiting the bandwidth of signals prior to digitization.

The highpass filter (HPF), figure 1b, is used to allow high frequency signals, above a cutoff frequency (f_p ,) to pass through the filter, while attenuating or stopping lower frequencies. The highpass filter is used to remove any DC (0 Hz) bias, attenuated AC power supply line artifacts, or otherwise reduce low frequencies that are not desired. Often lowpass and highpass filters are combined to form a bandpass filter.

The bandpass filter (BPF), figure 1c, only allows frequencies in a defined range to pass, above a lower cutoff frequency f_{pl} and below a higher frequency f_{ph} . All lower and higher frequency components are attenuated or stopped. Bandpass filters are commonly used in radio transmitters and receivers to select a desired range of frequencies, in telephones to simultaneously eliminate DC and low frequency interference while limiting the signal bandwidth, and as elements of audio equalizers to either enhance or limit frequency bands for recording or playback.

The bandstop filter (BSF), figure 1d, is used to attenuate or stop signals in a defined range, above a lower cutoff frequency f_{pl} and below a higher frequency f_{ph} . These filters are commonly used to remove signals in frequency bands that could easily interfere with desired signals or be strong enough to saturate amplifiers causing them to operate as non-linear devices. They are commonly used to remove AC power supply artifacts (from 50 to 60 Hz and harmonics) in medical or other electronics or to remove unwanted high power RF signal bands (citizen’s band radio or FM radio bands).

1.2 Real Filters Responses

When constructing real world filters, they do not have the ideal characteristics previously shown. The vertical transitions at the edges of filter passbands, commonly referred to as brick-wall filters, can not practically be constructed. Another concern is how smooth the filter is in the passband and stopband regions. Different circuit configurations cause the passband to have small variations in attenuation (referred to as the passband ripple), the stopband to have variations not larger than some value (referred to as the stopband ripple) or both effects to be present.

1.2.1 Transition Band

All filters require a transition region or band from the passband to the stopband as shown in Fig. 2 for a lowpass filter. The passband is defined from 0 Hz to f_p Hz, while the stopband begins at f_s Hz and extends to infinity. The existence of a transition band gives rise to the definition of a filter shape factor. The filter shape factor is defined as the ratio of the stopband bandwidth defined at a required filter attenuation divided by the passband bandwidth or

$$SF = \frac{BW_{SB}}{BW_{PB}}$$

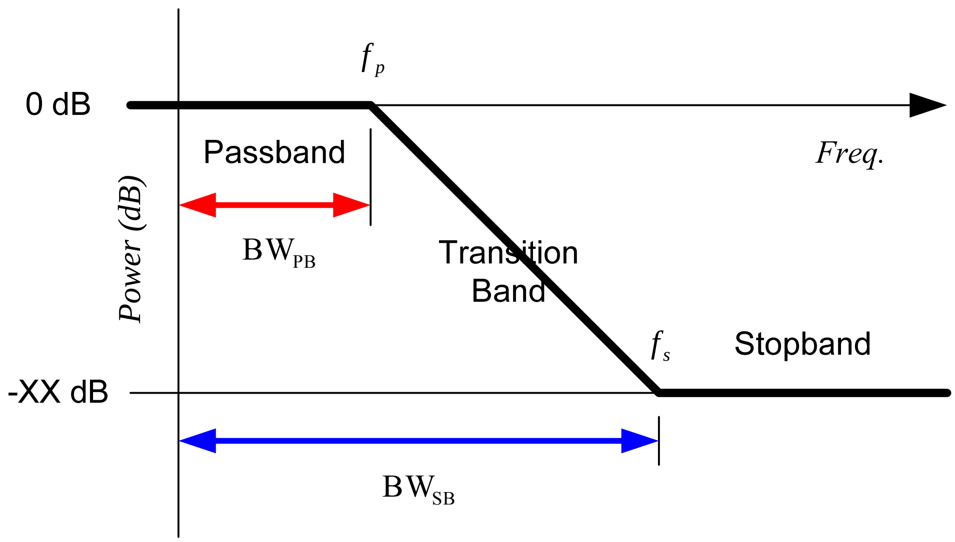


Figure 2: Lowpass Filter Transition Band and Shape Factor Definitions

For a bandpass filter, the practical transition bands and bandwidths used to define a shape factor are shown in Fig. 3. The passband extends from a lower to upper passband frequency, f_{pl} to f_{ph} Hz, while the stopband exists from 0 Hz to f_{sl} Hz and again from f_{su} Hz to infinity. As may be noted, for all practical LPF and BPF filters, the filter shape factor is greater than 1 since the stopband bandwidth is always wider than the passband bandwidth. In general, for shape factors greater than 3.3 simple RLC filters or R-C active filter can be employed, for shape factors from 1.5 to 3 more exotic crystal or SAW filter can be used, and for shape factors less than 1.5 an alternate design approach should be used.

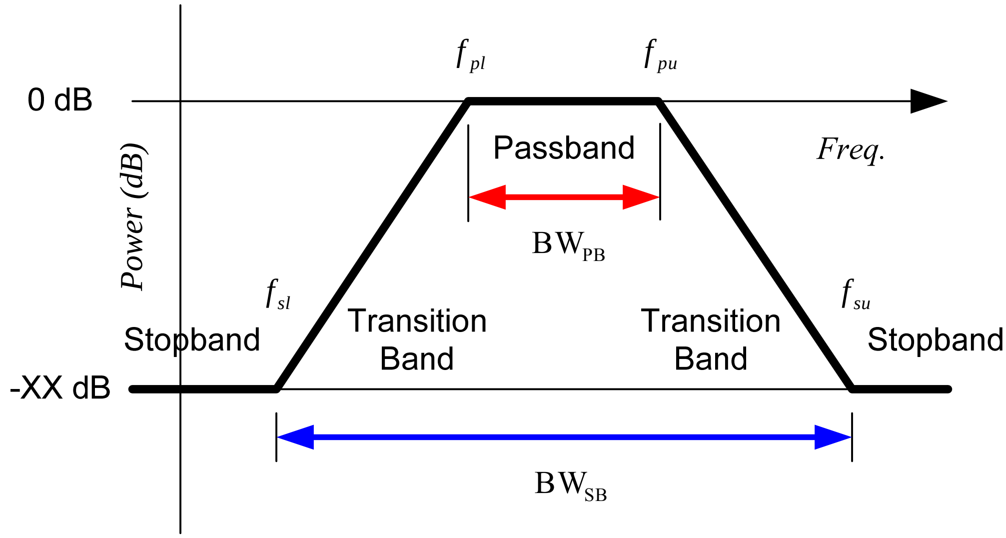


Figure 3: Bandpass Filter Transition Band and Shape Factor Definitions

As might be expected, the highpass filter and bandstop filters also have transitions bands. For the highpass filter the filter shape factor is undefined, while the bandstop filter uses the inverse of the equation shown to define the shape factor. The Bandstop transition band and bandwidth values are shown in Fig. 4.

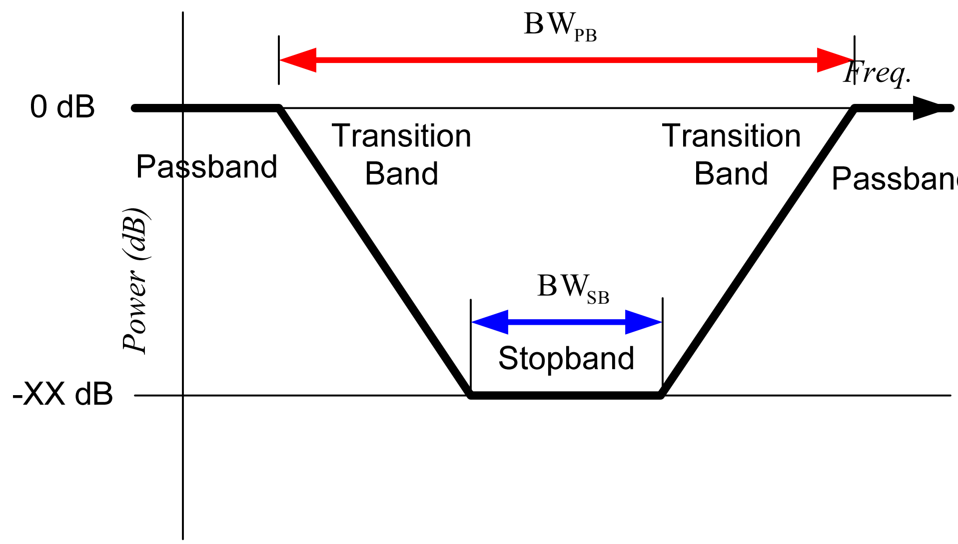


Figure 4: Bandstop Filter Transition Band and Shape Factor Definitions

The curves shown in this section are idealized, the passband has been normalized to zero dB or unity power gain and the passbands and stopbands are shown as perfectly smooth, flat lines. Normalization allows alternate filter approaches to be easily compared and separates voltage and power gain considerations from filter and spectral concerns. Meanwhile, the smooth curves in the various bands define the ideal filter response, when in fact the actual response may be significantly different. The following section shows how filters are typically specified and defines passband and stopband ripple.

1.2.2 Passband and Stopband Ripple

When defining a real filter, the shape of the power spectral response in the passband and stopband regions is important. Figure 5 shows a lowpass filter with both passband and stopband ripple. When defining a lowpass filter, typically the DC or 0 Hz value is set at 0 dB to insure that filtered signals have unity gain, thereby, applying no scaling to the DC bias point. Next, any allowable ripple, $\pm\delta_p$ in dB, from 0 Hz to the passband cutoff frequency f_p is defined. While numerous textbooks use a -3 dB power loss to define the lowpass filter cutoff frequency, f_p, practical RF and analog designs can not usually accept such a large ripple. A -3dB power point means that the signal is 50% of the power or 71% of the voltage! Therefore, it is important to understand and establish the allowable signal processing ripple for the passband. Typical RF designs use 0.5 or 1 dB ripple.

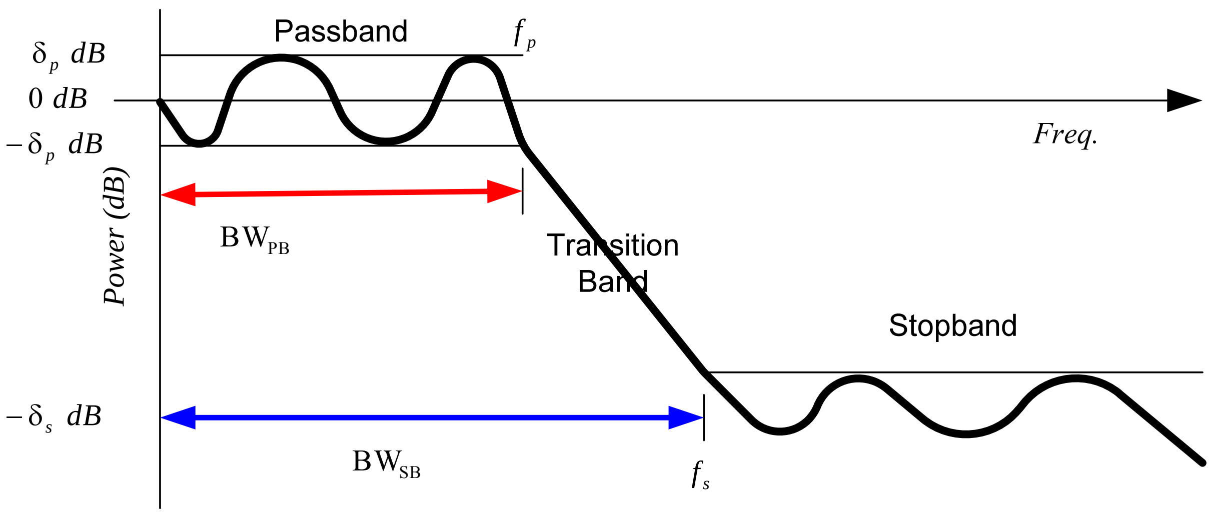


Figure 5: Lowpass Filter with Passband and Stopband Ripple

If the shape of the passband is important, the appropriate filter type must be selected. Butterworth and Chebyshev Type II filters have monotonic passbands, meaning that they are smooth curves that continuously drop off with no ripple. If a true ripple is allows, then Chebyshev Type I and Cauer/Elliptical filters can be used.

The stopband region is defined from f_s Hz to infinity by a maximum power level of $-\delta_s$ that can not be crossed. While many filters have monotonically decreasing power with frequency (e.g. -20 dB per decade of frequency for every transfer function pole), Chebyshev Type I or Elliptical filters may have one or more ripples that rise to the $-\delta_s$ power level.

An example of lowpass filter designs using MATLAB is shown in Fig. 6. The filter specification used was for a 1 kHz lowpass filter with an allowable 0.5 dB passband ripple and a 50 kHz stopband with a maximum -65 dB of stopband ripple. Four classic types of analog filters are shown along with lines showing the specification. The four filters are a Butterworth filter (monotonically decreasing passband, transition band and stopband), a Chebyshev Type I filter (passband ripple and monotonically decreasing stopband), a Chebyshev Type II filter (monotonically decreasing passband and stopband ripple), and an Elliptical or Cauer filter (passband and stopband ripple).

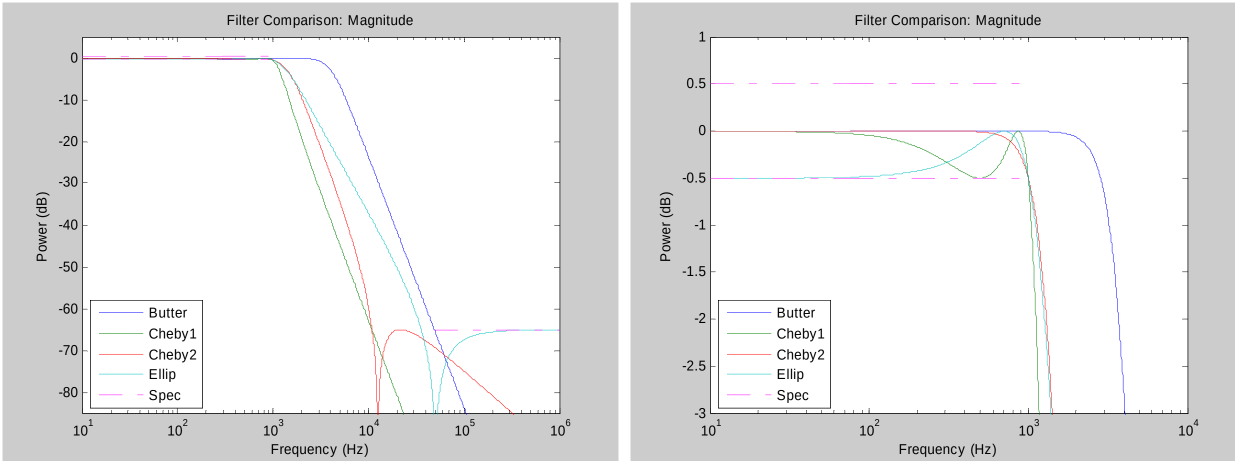


Figure 6: Example MATLAB Lowpass Filters

Highpass, bandpass and bandstop filters must be similarly defined for the allowable signal power gain (typically normalized), the passband cutoff frequencies and passband ripple, and the stopband frequencies and allowable ripple.

1.3 Filter Transfer Function and Spectral Response

The power spectral response of a filter transfer functions is used to describe filter performance as previously discussed. The following section briefly reviews transfer functions and the power spectrum of filters.

1.3.1 Transfer Functions

The transfer function of any system, particularly a filter can be describe in terms of a block diagram and equations as follows.

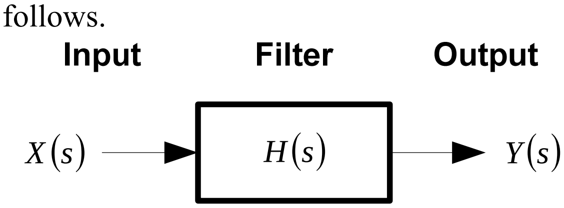


Figure 7: Filer Transfer Function Block Diagram

The filter transfer function is defined as.

$$\frac{Output}{Input} = H(s) = \frac{Y(s)}{X(s)}$$

or in terms of zeros, poles, and complex zero and pole pairs as

$$H(s) = K_{pz} \cdot \frac{\prod_{i=1}^{Q}(s+z_i)\cdot\prod_{v=1}^{V}\left(s^2+2\cdot\zeta_v\cdot w_v\cdot s+w_v^{\ 2}\right)}{s^R\cdot\prod_{m=1}^{M}(s+p_m)\cdot\prod_{n=1}^{N}\left(s^2+2\cdot\zeta_n\cdot w_n\cdot s+w_n^{\ 2}\right)}$$

1.3.2 Bode Plot

When analyzing for the frequency response, a Bode plot of the transfer function magnitude and phase is used

$$H(jw) = |H(jw)| \cdot \exp(\angle H(jw))$$

Returning to the transfer function and rearranging (to simplify for Bode plots)

$$H(s) = \left[K_{pz} \cdot \frac{\prod_{i=1}^{Q}(z_i) \cdot \prod_{v=1}^{V}\left(w_v^{\,2}\right)}{\prod_{m=1}^{M}(p_m) \cdot \prod_{n=1}^{N}\left(w_n^{\,2}\right)} \right] \cdot \frac{\prod_{i=1}^{Q}\left(1 + \frac{s}{z_i}\right) \cdot \prod_{v=1}^{V}\left(1 + 2 \cdot \zeta_v \cdot \frac{s}{w_v} + \left(\frac{s}{w_v}\right)^2\right)}{s^R \cdot \prod_{m=1}^{M}\left(1 + \frac{s}{p_m}\right) \cdot \prod_{n=1}^{N}\left(1 + 2 \cdot \zeta_n \cdot \frac{s}{w_n} + \left(\frac{s}{w_n}\right)^2\right)}$$

Using this form, the magnitude becomes (notice how easy it is to analyze at w = 0 and as w $\rightarrow \infty$)

$$|H(jw)| = K \cdot \frac{\prod_{i=1}^{Q}\sqrt{1 + \left(\frac{w}{z_i}\right)^2} \cdot \prod_{v=1}^{V}\sqrt{\left(1 - \left(\frac{w}{w_v}\right)^2\right)^2 + \left(2 \cdot \frac{\zeta_v \cdot w}{w_v}\right)^2}}{w^R \cdot \prod_{m=1}^{M}\sqrt{1 + \left(\frac{w}{p_m}\right)^2} \cdot \prod_{n=1}^{N}\sqrt{\left(1 - \left(\frac{w}{w_n}\right)^2\right)^2 + \left(2 \cdot \frac{\zeta_n \cdot w}{w_n}\right)^2}}$$

and the corresponding phase of the filter is :

$$\angle H(jw) = \sum_{i=1}^{Q} a\tan\left(\frac{w}{z_i}\right) + \sum_{v=1}^{V} a\tan\left(\frac{2 \cdot \frac{\zeta_v \cdot w}{w_v}}{1 - \left(\frac{w}{w_v}\right)^2}\right) - R \cdot \frac{\pi}{2} - \sum_{m=1}^{M} a\tan\left(\frac{w}{p_m}\right) - \sum_{n=1}^{N} a\tan\left(\frac{2 \cdot \frac{\zeta_n \cdot w}{w_n}}{1 - \left(\frac{w}{w_n}\right)^2}\right)$$

These are the equations plotted in a Bode Plot.

1.3.3 Power Spectrum

For most analysis, RF and some analog designers prefer to use the "power spectrum", where the power spectrum is the square of the real magnitude

$$PSD_H(jw) = |H(jw)|^2$$

As for any complex function, one means to form the magnitude is to take the square root of the function multiplied by the complex conjugate of the function. For s=jw this is

$$PSD_H(jw) = |H(jw)|^2 = H(jw) \cdot H(-jw)$$

Notice that there is no magnitude function or square root! When evaluating in the Laplace domain this becomes

$$PSD_H(jw) = H(s)\cdot H(-s) \qquad for\ s = jw$$

For filter design this is a great result! The power spectrum is purely real, but can be describe in terms of a transfer function, H(s), and the mirror image of the transfer function reflected about s=jw axis, H(-s). As such, we can always define a marginally stable filter realization of the power spectral density by "assigning" poles in the left-half plane of the s-domain to H(s) and poles in the right-half-plane to H(-s). Further, a minimum phase filter can also be guaranteed if all the zeros in the left-half plane are assigned to H(s) and all the zeros in the right-half plane are assigned to H(-s).

1.3.4 Filter Design Process

Filter may be designed or selected from a group of "classically defined" filters or arbitrarily based on the desired power spectral response curve.

Classically defined filters:

For lowpass, highpass, bandpass or bandstop filters explicit design classical procedures have been developed for a number of "named" filters, such as: Butterworth, Chebyshev, Inverse Chebyshev, and Cauer/Elliptical. A number of excellent textbooks [6] provide detailed descriptions of these filters. In addition, CAE tools are available on-line or using MATLAB. The MATLAB signal processing and filter toolboxes provide filter order estimation, filter design, and spectral plotting routines.

Arbitrary Filters:

If an arbitrary spectral response is required, the following steps should be taken:

(1) Define a smooth, continuous curve for the desired power spectrum of a filter. Try to use straight line segments from frequency to frequency, making sure to allow for transitions bands as you go (no brick walls!).

(2) Estimate the number and locations of poles and zeros based on the break-points and slope of the transitions bands.

(3) Use the pole and zero estimates in the following equation to see if the resulting curves are close enough

$$PSD_H(jw) = K^2 \cdot \frac{\prod_{i=1}^{Q}\left(1+\left(\frac{w}{z_i}\right)^2\right)\cdot\prod_{v=1}^{V}\left[\left(1-\left(\frac{w}{w_v}\right)^2\right)^2+\left(2\cdot\frac{\zeta_v\cdot w}{w_v}\right)^2\right]}{w^{2\cdot R}\cdot\prod_{m=1}^{M}\left(1+\left(\frac{w}{p_m}\right)^2\right)\cdot\prod_{n=1}^{N}\left[\left(1-\left(\frac{w}{w_n}\right)^2\right)^2+\left(2\cdot\frac{\zeta_n\cdot w}{w_n}\right)^2\right]}$$

(4) Iterate on your estimates until you like the curve.

(5) Once the poles and zeros are defined, use a sequential cascade of circuit stages to implement the transfer function. Each stage will typically perform one or two poles with or without one or two zeros if you are using op-amps (also called active circuit design).

Section 3.2 will show various active op-amp circuits that can be used as design stages.

2 Named Filters: Butterworth, Chebyshev and more

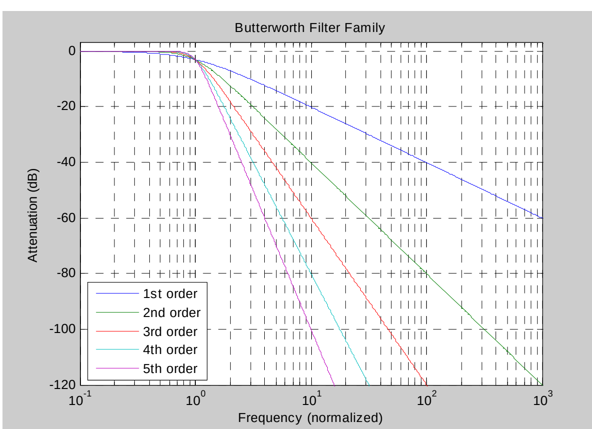
There are a number of well known filter families with desirable spectral properties in magnitude and phase. The primary families consist of:

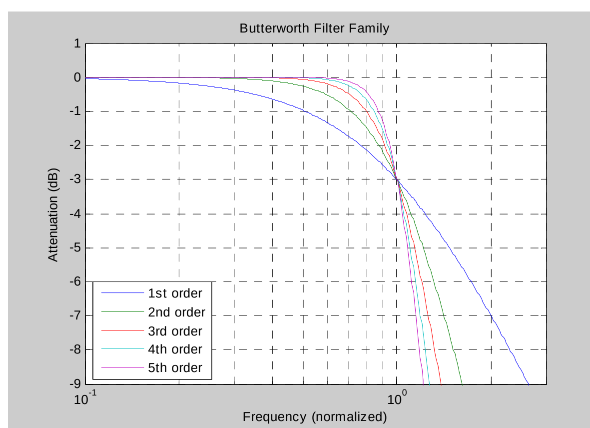
Butterworth Filters:	Monotonically decreasing magnitude, well defined ½ voltage (3 dB) cutoff frequency of the passband, predictable transition band, all poles analog filter
Chebychev Type I	Better transition band performance, ripple in the passband, ability to set ripple magnitude
Chebychev Type II	Ripple in the stopband, defined based on stop-band frequency,
Cauer or Elipical	Passband and stopband ripple

2.1 The Butterworth Lowpass Filter

$$T_n(jw)\cdot T_n(-jw)=\frac{1}{1+\left(w/w_0\right)^{2n}}$$

$$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(s/j\cdot w_0\right)^{2n}}=\frac{1}{1+(-j)^{2n}\cdot\left(s/w_0\right)^{2n}}=\frac{1}{1+(-1)^{n}\left(s/w_0\right)^{2n}}$$





Characteristic Eq. $$1+(-1)^{n}\left(s/w_0\right)^{2n}=\Delta(s)\cdot\Delta(-s)=0$$

Frequency normalized $$\Delta(s)\cdot\Delta(-s)=1+(-1)^{n}s^{2n}=0$$

An exceptional reference is [6], M.E. Van Valkenburg, Analog Filter Design, Oxford Univ. Press, 1982, ISBN: 0-19-510734-9

2.1.1 Solving for the Butterworth Filter poles:

Filter in jw
$$T_n(jw)\cdot T_n(-jw)=\frac{1}{1+\left(\frac{w}{w_0}\right)^{2n}}$$

Laplace
$$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(\frac{s}{j\cdot w_0}\right)^{2n}}=\frac{1}{1+(-j)^{2n}\cdot\left(\frac{s}{w_0}\right)^{2n}}=\frac{1}{1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}}$$

Characteristic Eq.
$$1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}=\Delta(s)\cdot\Delta(-s)=0$$

Normalize
$$\Delta(s)\cdot\Delta(-s)=1+(-1)^n s^{2n}=0$$

2.1.1.1 For n odd:

$$\Delta(s)\cdot\Delta(-s)=1-s^{2n}=\left(1+s^n\right)\cdot\left(1+(-s)^n\right)=0$$

Roots at
$$s^{2n}=1=\exp(j2\cdot m\cdot\pi)\rightarrow s=\exp\left(\frac{jm\cdot\pi}{n}\right)$$

Let $\Delta(s)$ be the LHP poles and $\Delta(-s)$ be the RHP poles

2.1.1.2 For n even:

$$\Delta(s)\cdot\Delta(-s)=1+s^{2n}=\left(1+js^n\right)\cdot\left(1-js^n\right)=0$$

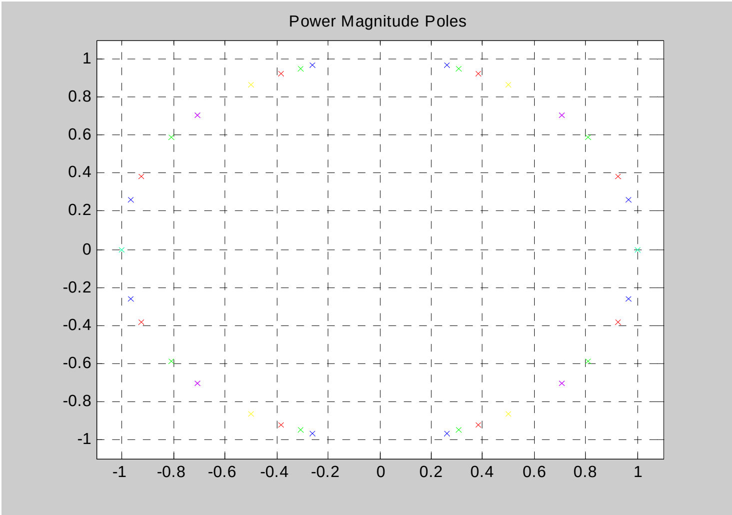
Roots at
$$s^{2n}=-1=\exp(j2\cdot m\cdot\pi+j\pi)\rightarrow s=\exp\left(\frac{j2\cdot m\cdot\pi+j\pi}{2\cdot n}\right)$$

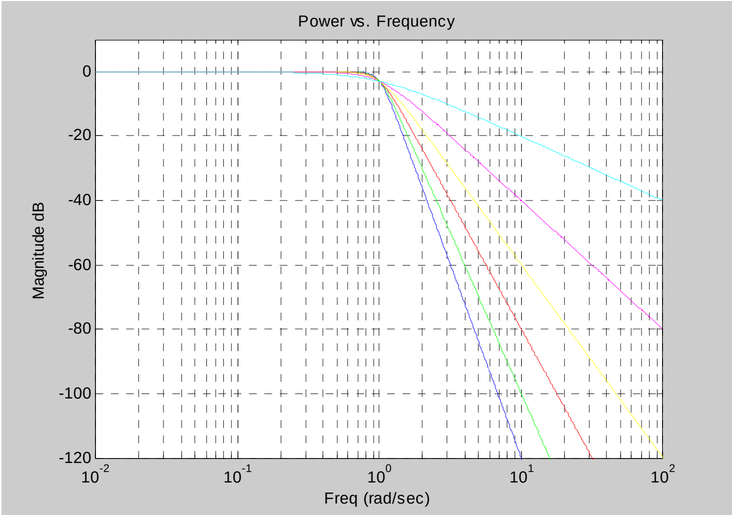
Let $\Delta(s)$ be the LHP poles and $\Delta(-s)$ be the RHP poles

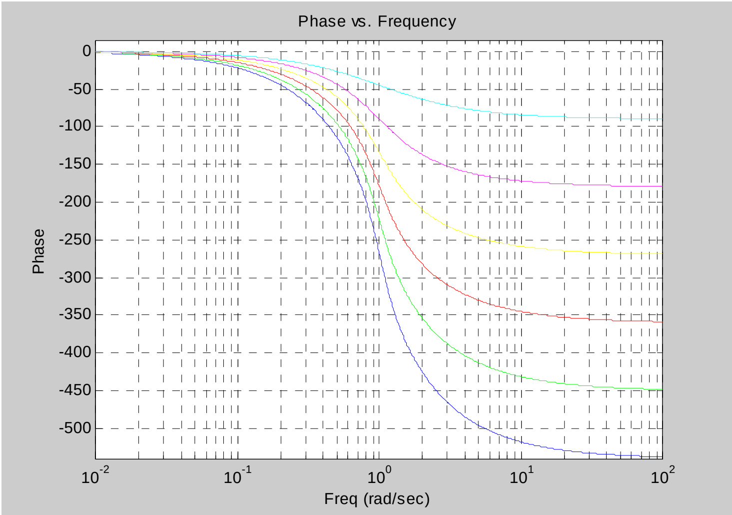
2.1.2 Matlab Code and Example Plots

```
% BW Filter generation demonstration
%

close all
clear all

Rin=1;
Rload=1;
Rmatch=1;

PBfreq=1;

PiW=logspace(log10(PBfreq)-2,log10(PBfreq)+2,1024);
```

```
colorseq=['b' 'g' 'r' 'y' 'm' 'c'];
ii=0;

PolesRange=6:-1:1

for BWn=PolesRange
ii=mod(ii,6)+1;

denP=roots([((-1/(PBfreq^2))^(BWn)) zeros(1,2*BWn-1) 1])
[Y,I] = sort(real(denP));
denPsort=denP(I)
den=poly(denPsort(1:BWn));

figure(1)
plot(real(denP),imag(denP),sprintf('%cx',colorseq(ii)) );
title('Power Magnitude Poles')
grid on;
hold on;

num = [PBfreq^(BWn)];

zpi=abs(roots(num));
ppi=abs(roots(den));
BWsys=tf(num,den)
[PiMAG, PiPHASE]=bode(BWsys,PiW);

figure(2)
semilogx(PiW, dBv(squeeze(PiMAG)),colorseq(ii) );
grid on;
hold on;
title('Power vs. Frequency')
xlabel('Freq (rad/sec)');
ylabel('Magnitude dB');
plotv=axis;
axis([plotv(1) plotv(2) -120 10])

figure(3)
semilogx(PiW, (squeeze(PiPHASE)),colorseq(ii) );
grid on;
hold on;
title('Phase vs. Frequency')
xlabel('Freq (rad/sec)');
ylabel('Phase');
axis([plotv(1) plotv(2) -max(PolesRange)*90 15])

pause
end
```


Power Magnitude Poles
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1


Power vs. Frequency
Magnitude dB
0
-20
-40
-60
-80
-100
-120
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}
Freq (rad/sec)


Phase vs. Frequency
Phase
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}
Freq (rad/sec)

2.1.3 What if we want to change the frequency …

$$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(\frac{s}{j\cdot w_0}\right)^{2n}}=\frac{1}{1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}}$$

Just change the natural frequency, $w_0 = 2\pi f_0$; the center frequency is simply scaled!

$$T_n(s)\cdot T_n(-s)=\frac{w_0^{\;2n}}{w_0^{\;2n}+(-1)^n(s)^{2n}}$$

Design approach:

1. Determine the order of the filter you want. What attenuation do you need at the $10\cdot w_0$ point?
(There are plenty of curves, like those above, if the value you need comes before t 10x the cutoff frequency.)

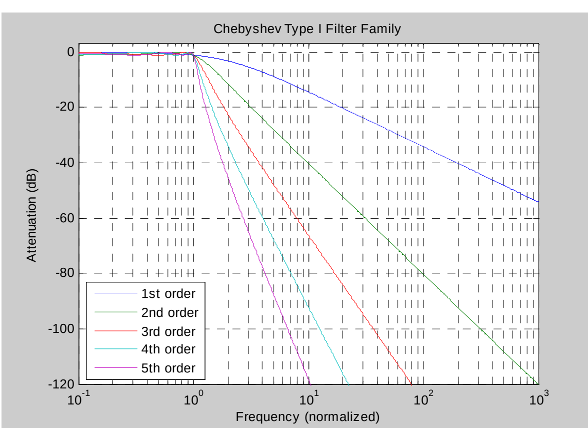
2. Generate the Butterworth Coefficients on the unit circle for w=1

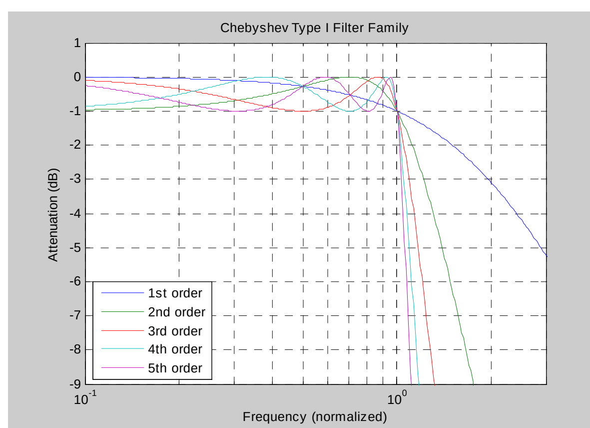
3. Scale the poles by the desired frequency (remember that w=1 is in radians/sec, therefore multiply by $w_0 = 2\pi f_0$.

2.2 *The Chebyshev Lowpass Filter*

$$\mathrm{T_n}(jw)\cdot \mathrm{T_n}(-jw)=\frac{1}{1+\varepsilon^2\cdot \mathrm{C_n}\left(\frac{w}{w_0}\right)^2}$$

where $$\mathrm{C_n}(w)=\cos(n)\cdot\cos^{-1}(w)$$

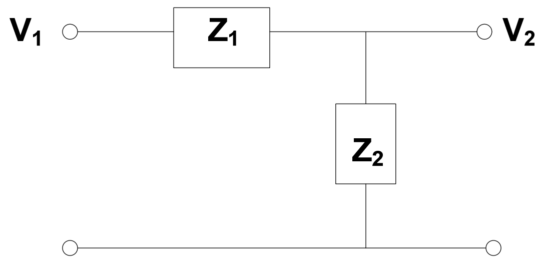




3 Simple Circuits that Perform Filtering

3.1 Passive "Impedance Divider" Filter Circuit

To see how complex impedances are used in practice, consider the simple case of a voltage divider.



$$\frac{V_o}{V_i} = \frac{Z_2}{Z_1 + Z_2}$$

3.1.1 Low Pass Filter

If Z_1 is a resistor and Z_2 is a capacitor then

$$\frac{V_o}{V_i} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + sCR}$$

Generally we will be interested only in the magnitude of the response:

$$\left|\frac{V_o}{V_i}\right| = \left|\frac{1}{1 + sCR}\right| = \left|\frac{1}{1 + j\omega CR}\right| = \frac{1}{\sqrt{1^2 + (\omega CR)^2}}$$

Recall from ECE 210 that the magnitude of a complex number is the square root of the sum of the squares of the real and imaginary parts. There are also phase shifts associated with the transfer function (or gain, V_o/V_i), though we will generally ignore these.

This is obviously a low pass filter (i.e., low frequency signals are passed and high frequency signals are blocked). If $\omega << 1/RC$ then $\omega RC << 1$ and the magnitude of the gain is approximately unity, and the output equals the input. If $\omega >> 1/RC$ ($\omega RC >> 1$)then the gain goes to zero, as does the output. At $\omega = 1/RC$, called the break frequency (or cutoff frequency, or 3dB frequency, or half-power frequency, or bandwidth), the magnitude of the gain is $1/\sqrt{2} \cong 0.71$. In this case (and all first order RC circuits) high frequency is defined as

$\omega >> 1/RC$; the capacitor acts as a short circuit and all the voltage is across the resistance. At low frequencies, $\omega << 1/RC$, the capacitor acts as an open circuit and there is no current (so the voltage across the resistor is near zero).

If Z_1 is an inductor and Z_2 is a resistor another low pass structure results with a break frequency of R/L.

3.1.2 A Simple High-Pass Circuit

If Z_1 is a capacitor and Z_2 is a resistor we can repeat the calculation:

$$\frac{V_o}{V_i} = \frac{R}{\frac{1}{sC} + R} = \frac{sCR}{1 + sCR}$$

and

$$\left|\frac{V_o}{V_i}\right| = \left|\frac{sCR}{1 + sCR}\right| = \left|\frac{j\omega CR}{1 + j\omega CR}\right| = \frac{\omega CR}{\sqrt{1^2 + (\omega CR)^2}}$$

At high frequencies, $\omega >> 1/RC$, the capacitor acts as a short and the gain is 1 (the signal is passed). At low frequencies, $\omega << 1/RC$, the capacitor is an open and the output is zero (the signal is blocked). This is obviously a high pass structure and you can show that the break frequency is again $1/RC$.

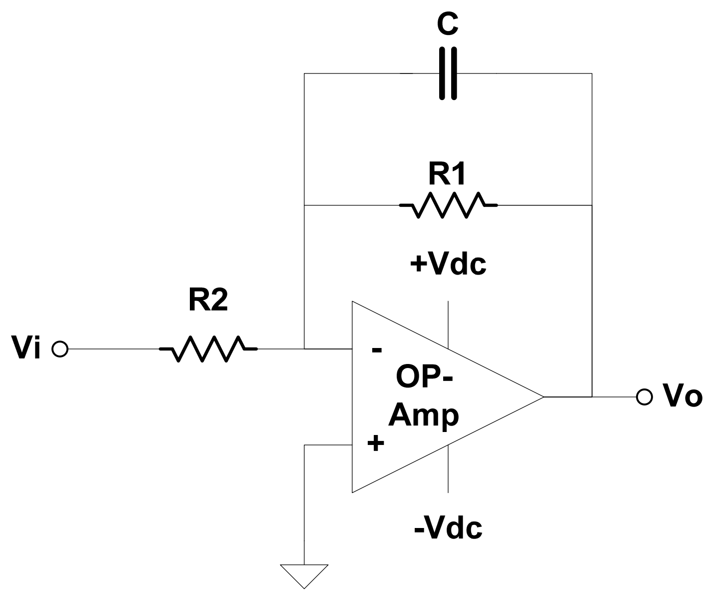
If Z_1 is a resistor and Z_2 is an inductor the resulting circuit is high pass with a break frequency of R/L.

This concept of a complex impedance is extremely powerful and can be used when analyzing operational amplifier circuits, as you will soon see.

3.2 Simple Active Filters with Op Amps

3.2.1 Low-Pass filters - the integrator reconsidered.

In this section we study the frequency response of an op-amp integrator.



First Order Low-pass Filter with Op Amp

If you derive the transfer function for the circuit above you will find that it is of the form:

$$\frac{V_o}{V_i} = \frac{H_o \omega_o}{s + \omega_o}, \text{ or } \left|\frac{V_o}{V_i}\right| = |H_o| \frac{\omega_o}{\sqrt{\omega^2 + \omega_o^2}}$$

Derivation:

Using nodal analysis:

$$\frac{V_i - V_p}{R_2} = \frac{V_p - V_{out}}{\frac{R_1}{sC} \Big/ \left(R_1 + \frac{1}{sC}\right)}$$

Where $V_p = 0$, and is the voltage (virtual ground) at the non-inverting terminal

$$\frac{-V_{out}}{V_i} = \frac{\frac{R_1}{sC} \Big/ \left(R_1 + \frac{1}{sC}\right)}{R_2}$$

$$\frac{V_{out}}{V_i} = \frac{-\left(R_1 / (sR_1C + 1) \right)}{R_2}$$

Therefore,

$$\frac{V_{out}}{V_i} = \frac{-R_1}{R_2} \cdot \frac{1}{1 + sR_1C} = H_o \cdot \frac{1}{1 + s/\omega_o}$$

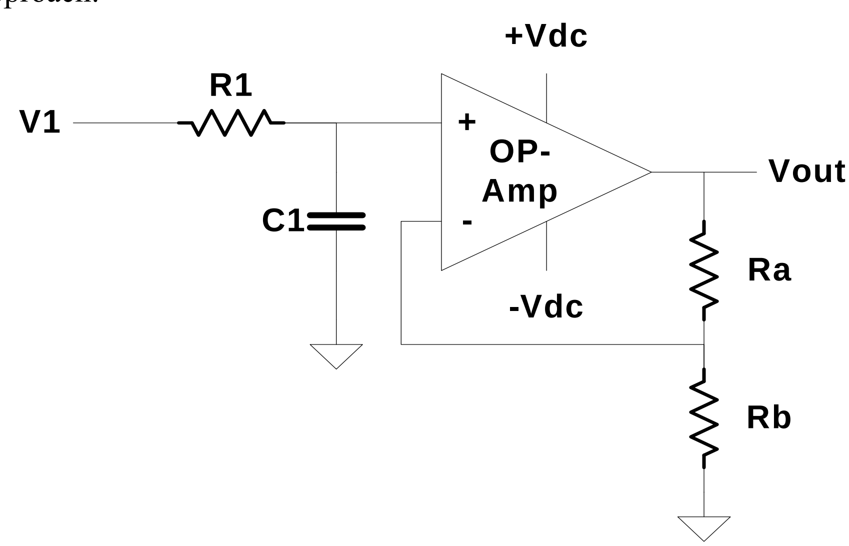
So,

$$MaxGain = G = H_o = -\frac{R_1}{R_2} \qquad \omega_0 = \frac{1}{R_1C}$$

which is the general form for first-order (one reactive element) low-pass filters. At high frequencies ($\omega >> \omega_o$) the capacitor acts as a short, so the gain of the amplifier goes to zero. At very low frequencies ($\omega << \omega_o$) the capacitor is an open and the gain of the circuit is H_o. But what do we mean by low (or high) frequency?

We can consider the frequency to be high when the large majority of current goes through the capacitor; i.e., when the magnitude of the capacitor impedance is much less than that of R_1. In other words, we have high frequency when $1/\omega C << R_1$, or $\omega >> 1/R_1C = \omega_o$. Since R_1 now has little effect on the circuit, it should act as an integrator. Likewise low frequency occurs when $\omega << 1/R_1C$, and the circuit will act as an amplifier with gain $-R_1/R_2 = H_o$.

An alternate approach:



Derivation:

Using circuit analysis techniques – in this case, nodal analysis:

$$\frac{V_1 - V_p}{R_1} = \frac{V_p}{1/sC_1}$$

where V_p is the voltage at the non-inverting terminal

$$V_p\left(\frac{1}{R_1} + sC_1\right) = \frac{V_1}{R_1}$$

$$V_n = \frac{R_b}{R_a + R_b} \cdot V_{out} \qquad \text{voltage divider}$$

where V_n is the voltage at the inverting terminal

Let $V_p = V_n$

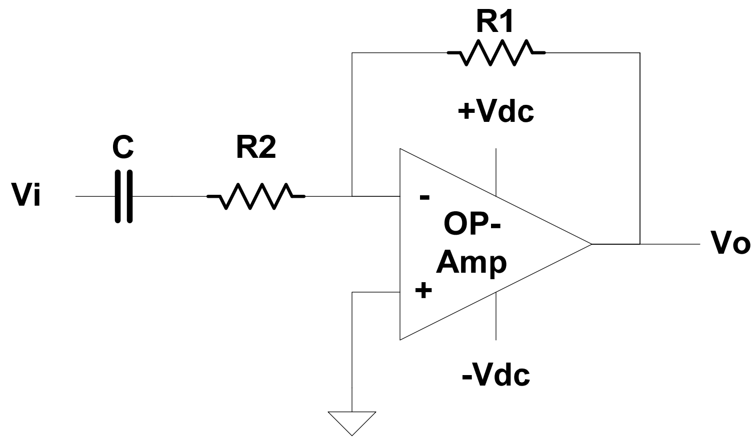
$$V_{out} \cdot \frac{R_b}{R_a + R_b} \cdot \left(\frac{1}{R_1} + sC_1\right) = \frac{V_1}{R_1}$$

$$\frac{Vout(s)}{Vin(s)} = \frac{R_a + R_b}{R_b} \cdot \frac{1}{1 + sR_1C_1}$$

$$MaxGain = G = \frac{R_a + R_b}{R_b} \qquad \omega_0 = \frac{1}{R_1C_1}$$

3.2.2 High-Pass filters - the differentiator reconsidered.

The circuit below is a modified differentiator, and acts as a high pass filter.



First Order High Pass Filter with Op Amp

Using Nodal analysis (ECE 2100):

$$\frac{V_i - 0}{1/sC + R_2} = \frac{0 - V_o}{R_1}$$

where "0" is the virtual ground at the inverting terminal of the op-amp

This gives:

$$-\frac{V_o}{V_i} = \frac{sR_1C}{1 + sR_2C}$$

Which results in:

$$\frac{V_o}{V_i} = -R_1C \cdot \frac{s}{1 + sR_2C} = \frac{-R_1}{R_2} \cdot \frac{s}{1/R_2C + s}$$

Using analysis techniques similar to those used for the low pass filter, it can be shown that

$$\frac{V_o}{V_i} = H_o \frac{s}{s + \omega_o}, \text{ or } \left|\frac{V_o}{V_i}\right| = |H_o| \frac{\omega}{\sqrt{\omega^2 + \omega_o^2}}$$

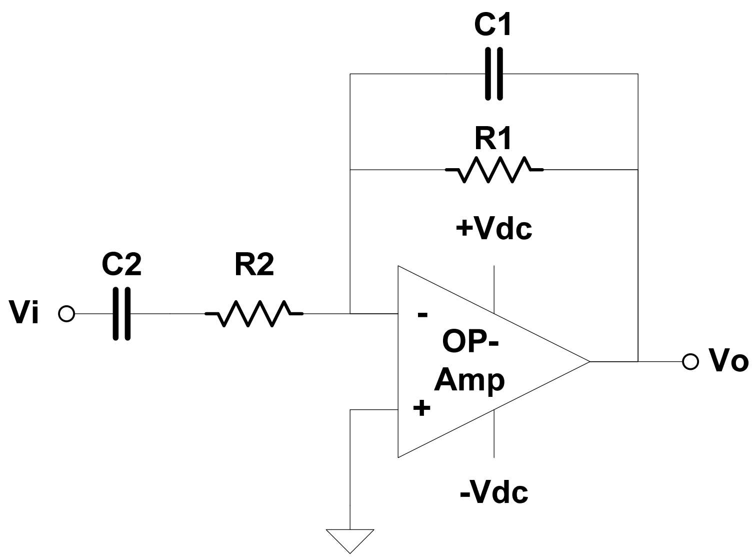
which is the general form for first-order (one reactive element) high-pass filters. At high frequencies ($\omega >> \omega_o$) the capacitor acts as a short, so the gain of the amplifier goes to

$-R_1/R_2 = H_o$.At very low frequencies ($\omega<<\omega_o$) the capacitor is an open and the gain of the circuit is H_o. For this circuit $\omega_o = 1/R_2C$.Therefore this circuit is a high-pass filter (it passes high frequency signals, and blocks low frequency signals.

3.2.3 Band-Pass circuits

Besides low-pass filters, other common types are high-pass (passes only high frequency signals), band-reject (blocks certain signals) and band-pass (rejects high and low frequencies, passing only signal around some intermediate frequency).

The simplest band-pass filter can be made by combining the first order low pass and high pass filters that we just looked at.



Simple Band Pass Filter with Op Amp

Using Nodal analysis (ECE 210):

$$\frac{V_i - 0}{1/sC_2 + R_2} = \frac{0 - V_o}{\frac{R_1}{sC_1} \Big/ \left(R_1 + \frac{1}{sC_1}\right)}$$

where “0” is the virtual ground at the inverting terminal of the op-amp

This gives:

$$\frac{V_i}{sC_2/(sC_2R_2 + 1)} = -\frac{V_o}{R_1/(sC_1R_1 + 1)}$$

$$\frac{R_1 / (sC_1R_1+1)}{sC_2 / (sC_2R_2+1)} = -\frac{V_o}{V_i}$$

$$\frac{V_o}{V_i} = -\frac{R_1}{R_2}\frac{sC_2R_2}{s^2(C_1R_1C_2R_2)+s(C_1R_1+C_2R_2)+1}$$

Which is of the form

$$H(s) = \frac{V_o}{V_i} = \frac{H_o\beta s}{s^2+\beta s+\omega_o^2}$$

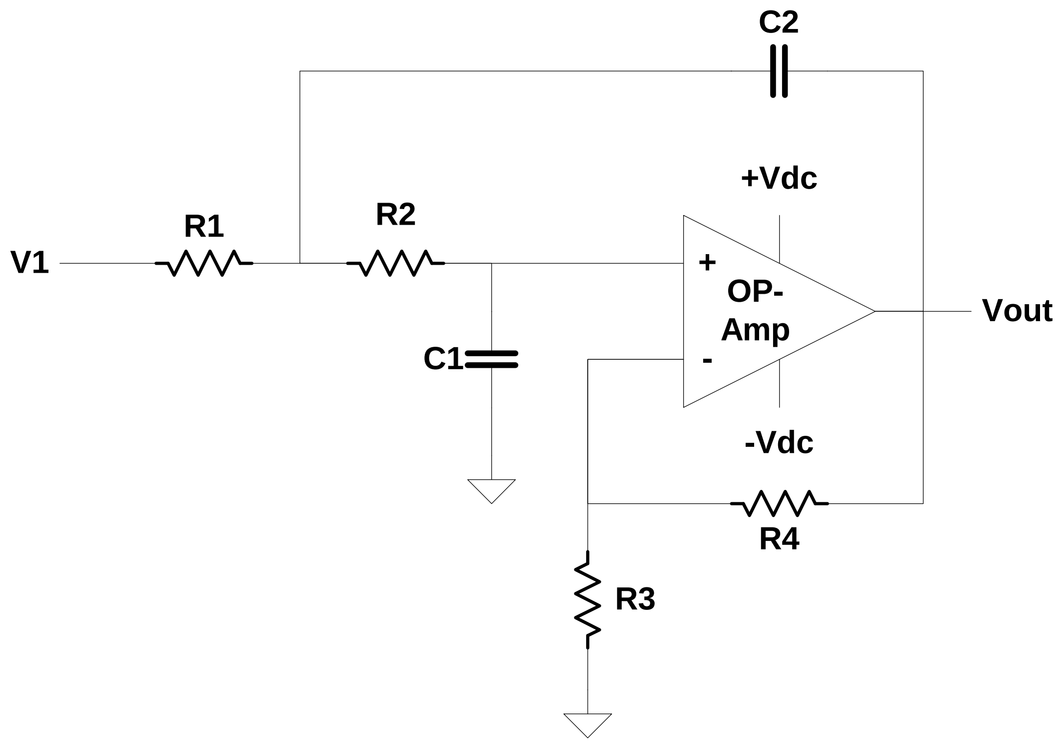
Where $\beta = \frac{C_1R_1+C_2R_2}{C_1R_1C_2R_2}$ and $\omega_o = \frac{1}{\sqrt{C_1R_1C_2R_2}}$

This circuit will attenuate low frequencies $\left(\omega << 1/R_2C_2\right)$ and high frequencies $\left(\omega >> 1/R_1C_1\right)$, but will pass intermediate frequencies with a gain of $-R_1/R_2$ .However, this circuit cannot be used to make a filter with a very narrow band. To do that requires a more complex filter as discussed below.

3.3 Second Order Active Filters with Op Amps

3.3.1 Sallen-Key Circuit Lowpass Filter

An active lowpass filter implementation of a unity gain Friend Circuit, also referred to as a Sallen-Key circuit as described in [4]: Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapoli, IN, 1974.



The transfer function for this circuit is

$$\frac{Vout(s)}{V1(s)} = \frac{\left(\frac{R3+R4}{R3}\right)\cdot\frac{1}{C1\cdot C2\cdot R1\cdot R2}}{s^2 + s\cdot\left(\frac{1}{C2\cdot R1}+\frac{1}{C2\cdot R2}-\frac{R4/R3}{C1\cdot R2}\right)+\frac{1}{C1\cdot C2\cdot R1\cdot R2}} = \frac{K\cdot w^2}{s^2+2\cdot\zeta\cdot w\cdot s+w^2}$$

$$\frac{Vout(s)}{V1(s)} = \left(\frac{R3+R4}{R3}\right)\cdot\frac{1}{1+s\cdot\left(C1\cdot R2+C1\cdot R1-R4/R3\cdot C2\cdot R1\right)+\frac{s^2}{C1\cdot C2\cdot R1\cdot R2}}$$

Letting $C1 = C2 = C$ and $R1 = R2 = R$ and $G = \frac{R3+R4}{R3}$

$$MaxGain = G = \frac{R3+R4}{R3} \qquad w_0 = \frac{1}{C\cdot R}$$

and $\zeta = \frac{3-G}{2}$

3.3.1.1 Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C2 \right) = \frac{V1}{R1} + \frac{Vp}{R2} + Vo \cdot s \cdot C2$$

$$Vp \cdot \left(\frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R2}$$

$$Vn = \frac{R3}{R3 + R4} \cdot Vo$$

Letting $$Vp = Vn$$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R2}$$

$$V2 = \frac{R3 \cdot (1 + s \cdot C1 \cdot R2)}{R3 + R4} \cdot Vo$$

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C2 \right) = \frac{V1}{R1} + \frac{V2}{R2 \cdot (1 + s \cdot C1 \cdot R2)} + Vo \cdot s \cdot C2$$

$$V2 \cdot \left(\frac{1}{R1} + \frac{s \cdot C1}{1 + s \cdot C1 \cdot R2} + s \cdot C2 \right) = \frac{V1}{R1} + Vo \cdot s \cdot C2$$

$$Vo \cdot \frac{R3 \cdot (1 + s \cdot C1 \cdot R2)}{R3 + R4} \cdot \left(\frac{1}{R1} + \frac{s \cdot C1}{1 + s \cdot C1 \cdot R2} + s \cdot C2 \right) - Vo \cdot s \cdot C2 = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 + s \cdot C2 \cdot R1 + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}{R1} \right) - Vo \cdot s \cdot C2 = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 - s \cdot C2 \cdot R1 \cdot \left(\frac{R4}{R3} \right) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}{R1} \right) = \frac{V1}{R1}$$

$$Vo = \left(\frac{R3 + R4}{R3}\right) \cdot \left(\frac{1}{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 - s \cdot C2 \cdot R1 \cdot \left(\frac{R4}{R3}\right) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}\right) \cdot V1$$

$$\frac{Vo}{V1} = \frac{\left(\frac{R3 + R4}{R3}\right) \cdot \frac{1}{C1 \cdot C2 \cdot R1 \cdot R2}}{s^2 + s \cdot \left(\frac{1}{C2 \cdot R1} + \frac{1}{C2 \cdot R2} - \frac{R4/R3}{C1 \cdot R2}\right) + \frac{1}{C1 \cdot C2 \cdot R1 \cdot R2}}$$

Letting $C1 = C2 = C$ and $R1 = R2 = R$ and $G = \frac{R3 + R4}{R3}$

$$\frac{Vo}{V1} = \frac{G \cdot \frac{1}{(C \cdot R)^2}}{s^2 + s \cdot \left(\frac{3 - G}{C \cdot R}\right) + \frac{1}{(C \cdot R)^2}}$$

Resulting in

$$MaxGain = G = \frac{R3 + R4}{R3} \qquad w_0 = \frac{1}{C \cdot R}$$

And $\zeta = \frac{3 - G}{2}$

Note that for a stable system $1 \le G < 3$

Implying that $0 \le R4 < 2 \cdot R3$

3.3.2 Multiple Feedback Lowpass Filter

An active lowpass filter implementation of a unity gain multiple feedback implementation. This circuit has also been referred to as a Friend Circuit, and as a Sallen-Key circuit as described in [10].

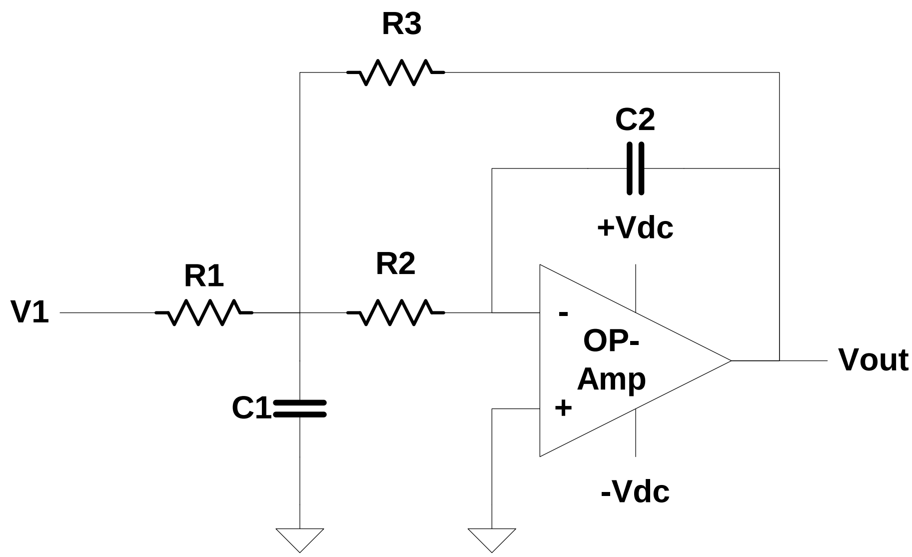


Figure 8. Multiple Feedback Lowpass Filter

The transfer function for this circuit is

$$\frac{Vo}{V1} = -\frac{1/C1 \cdot C2 \cdot R1 \cdot R2}{s^2 + s \cdot \left(1/C1 \cdot R1 + 1/C1 \cdot R2 + 1/C1 \cdot R3\right) + 1/C1 \cdot C2 \cdot R2 \cdot R3}$$

A unity gain version of the filter uses the following:

Letting $C_2 = C$, $C_1 = n \cdot C$, $R_1 = R_3 = R$ and $R_2 = m \cdot R$

Resulting in

$$MaxGain = G = -\frac{R_3}{R_1} = -1 \qquad Q = \sqrt{m \cdot n}/2 \cdot m + 1 \qquad w_0 = 1/C \cdot R \cdot \sqrt{m \cdot n}$$

If filter gain is desired use:

Letting $C_2 = C$, $C_1 = n \cdot C$, $R_1 = R_2 = R$ and $R_3 = m \cdot R$

Resulting in

$$MaxGain = G = -\frac{R_3}{R_1} = -m \qquad Q = \sqrt{m \cdot n}/2 \cdot m + 1 \qquad w_0 = 1/C \cdot R \cdot \sqrt{m \cdot n}$$

3.3.2.1 Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + s \cdot C1 \right) = \frac{V1}{R1} + \frac{Vo}{R3}$$

$$s \cdot C2 \cdot Vo + \frac{V2}{R2} = 0$$

Combining $Vo \cdot (-s \cdot C2 \cdot R2) \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + s \cdot C1 \right) = \frac{V1}{R1} + \frac{Vo}{R3}$

$$Vo \cdot (-s \cdot C2 \cdot R2) \cdot \left(\frac{R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3 + s \cdot C1 \cdot R1 \cdot R2 \cdot R3}{R1 \cdot R2 \cdot R3} \right) - \frac{Vo}{R3} = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R1 + s \cdot C2 \cdot (R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3}{R1 \cdot R3} \right) = -\frac{V1}{R1}$$

$$\frac{Vo}{V1} = -\frac{R3}{R1 + s \cdot C2 \cdot (R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3}$$

$$\frac{Vo}{V1} = -\frac{1/C1 \cdot C2 \cdot R1 \cdot R2}{s^2 + s \cdot \left(1/C1 \cdot R1 + 1/C1 \cdot R2 + 1/C1 \cdot R3 \right) + 1/C1 \cdot C2 \cdot R2 \cdot R3}$$

Letting $C_2 = C$, $C_1 = n \cdot C$, $R_1 = R_2 = R$ and $R_3 = m \cdot R$

$$\frac{Vo}{V1} = -\frac{\frac{1}{n \cdot (C \cdot R)^2}}{s^2 + s \cdot \left(\frac{1}{C \cdot R} \right) \cdot \left(\frac{2 \cdot m + 1}{m \cdot n \cdot} \right) + \frac{1}{n \cdot m \cdot (C \cdot R)^2}}$$

Resulting in

$$MaxGain = G = -\frac{R_3}{R_1} = -m \quad Q = \sqrt{m \cdot n} / 2 \cdot m + 1 \quad w_0 = 1 / C \cdot R \cdot \sqrt{m \cdot n}$$

An alternate unity gain configuration also exists.

Letting $C_2 = C$, $C_1 = n \cdot C$, $R_1 = R_3 = R$ and $R_2 = m \cdot R$

$$\frac{Vo}{V1} = -\frac{\frac{1}{n \cdot m \cdot (C \cdot R)^2}}{s^2 + s \cdot \left(\frac{1}{C \cdot R}\right) \cdot \left(\frac{2 \cdot m + 1}{m \cdot n \cdot}\right) + \frac{1}{n \cdot m \cdot (C \cdot R)^2}}$$

Resulting in

$$MaxGain = G = -\frac{R_3}{R_1} = -1 \qquad Q = \sqrt{m \cdot n} \Big/ 2 \cdot m + 1 \qquad w_0 = 1 \Big/ C \cdot R \cdot \sqrt{m \cdot n}$$

3.3.3 Third-Order Sallen-Key Variant Circuit Lowpass Filter

An active lowpass filter implementation based on an EDN Design Idea article [11].

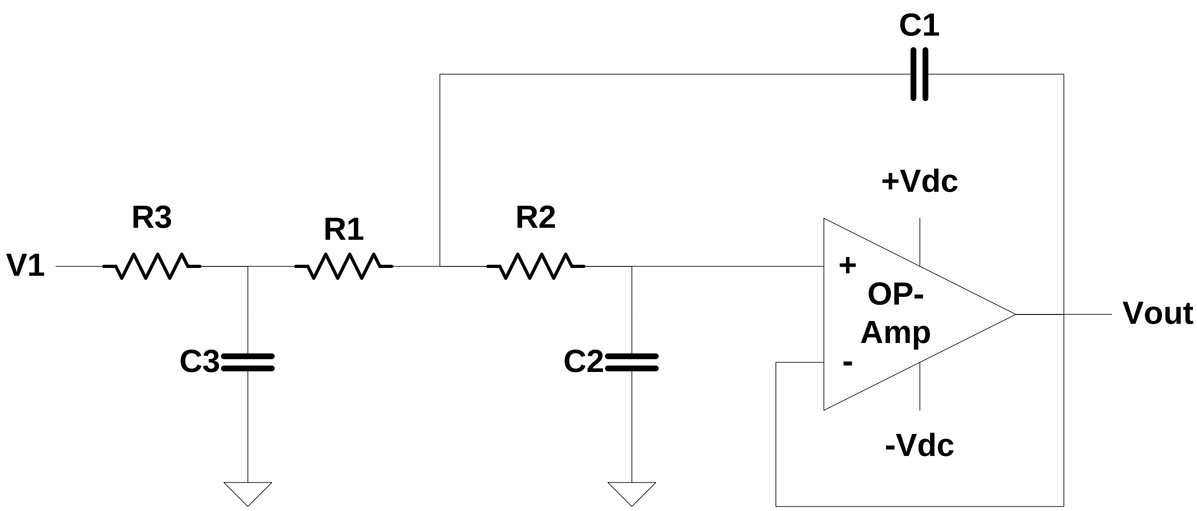


Figure 9. EDN Sallen-Key Lowpass Filter [11]

The transfer function for this circuit is

Letting $R1 = R2 = R3 = R$

$$\frac{Vo}{V1} = \frac{\frac{1}{R^3 \cdot (C1 \cdot C2 \cdot C3)}}{s^3 + s^2 \cdot \frac{2}{R \cdot C3} + s \cdot \frac{(C3 + 3 \cdot C2)}{R^2 \cdot (C1 \cdot C2 \cdot C3)} + \frac{1}{R^3 \cdot (C1 \cdot C2 \cdot C3)}}$$

$$MaxGain = G = 1 \qquad w_0 = 1 \Big/ R^3 \cdot (C1 \cdot C2 \cdot C3)$$

Note: this produce three poles, determined by the values of the capacitors when equal resistors are used.

3.3.3.1 Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$Vo = V+$$

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} + s \cdot C3 \right) = \frac{V1}{R3} + \frac{V3}{R1}$$

$$V3 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R1} + Vo \cdot \left(\frac{1}{R2} + s \cdot C1 \right)$$

$$Vo \cdot \left(\frac{1}{R2} + s \cdot C2 \right) = \frac{V3}{R2}$$

Solving for V3 in terms of Vo

$$Vo \cdot (1 + s \cdot R2 \cdot C2) = V3$$

Using V3
$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} + s \cdot C3 \right) = \frac{V1}{R3} + \frac{Vo \cdot (1 + s \cdot R2 \cdot C2)}{R1}$$

rearranging
$$V2 = \frac{R1 \cdot R3}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C3} \cdot \left[\frac{V1}{R3} + \frac{Vo \cdot (1 + s \cdot R2 \cdot C2)}{R1} \right]$$

$$V2 = \frac{V1 \cdot R1 + Vo \cdot R3 \cdot (1 + s \cdot R2 \cdot C2)}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C3}$$

Again using V3
$$Vo \cdot (1 + s \cdot R2 \cdot C2) \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R1} + Vo \cdot \left(\frac{1 + s \cdot R2 \cdot C1}{R2} \right)$$

rearranging
$$Vo \cdot \left[\frac{(1 + s \cdot R2 \cdot C2) \cdot (R1 + R2 + s \cdot R1 \cdot R2 \cdot C1) - R1 \cdot (1 + s \cdot R2 \cdot C1)}{R2} \right] = V2$$

$$Vo \cdot \left[\frac{R2 + s \cdot \left[R2^2 \cdot C2 + R1 \cdot R2 \cdot C2 \right] + s^2 \cdot R1 \cdot R2^2 \cdot C1 \cdot C2}{R2} \right] = V2$$

$$Vo \cdot \left[1 + s \cdot \left[R2 \cdot C2 + R1 \cdot C2 \right] + s^2 \cdot R1 \cdot R2 \cdot C1 \cdot C2 \right] = V2$$

Combining

$$Vo\cdot\left[1+s\cdot\left[R2\cdot C2+R1\cdot C2\right]+s^2\cdot R1\cdot R2\cdot C1\cdot C2\right]=\frac{V1\cdot R1+Vo\cdot R3\cdot(1+s\cdot R2\cdot C2)}{R1+R3+s\cdot R1\cdot R3\cdot C3}$$

$$Vo\cdot\left[\begin{array}{l}R1+R3+\\ +s\cdot\left(R1\cdot R3\cdot C3+R1\cdot R2\cdot C2+R1^2\cdot C2+R2\cdot R3\cdot C2+R1\cdot R3\cdot C2\right)\\ +s^2\cdot\left(R1^2\cdot R2\cdot C1\cdot C2+R1\cdot R2\cdot R3\cdot C1\cdot C2\right)\\ +s^3\cdot\left(R1^2\cdot R2\cdot R3\cdot C1\cdot C2\cdot C3\right)\\ -R3-s\cdot R2\cdot R3\cdot C2\end{array}\right]=V1\cdot R1$$

$$\frac{Vo}{V1}=\frac{1}{\begin{array}{l}1+s\cdot(R3\cdot C3+R2\cdot C2+R1\cdot C2+R3\cdot C2)+s^2\cdot(R1\cdot R2\cdot C1\cdot C2+R2\cdot R3\cdot C1\cdot C2)\\ +s^3\cdot(R1\cdot R2\cdot R3\cdot C1\cdot C2\cdot C3)\end{array}}$$

$$\frac{Vo}{V1}=\frac{1}{\begin{array}{l}1+s\cdot(R3\cdot C3+(R1+R2+R3)\cdot C2)+s^2\cdot R2\cdot(R1+R3)\cdot C1\cdot C2\\ +s^3\cdot(R1\cdot R2\cdot R3\cdot C1\cdot C2\cdot C3)\end{array}}$$

Let R1=R2=R3=R

$$\frac{Vo}{V1}=\frac{1}{1+s\cdot R\cdot(C3+3\cdot C2)+2\cdot s^2\cdot R^2\cdot C1\cdot C2+s^3\cdot R^3\cdot(C1\cdot C2\cdot C3)}$$

$$\frac{Vo}{V1}=\frac{\frac{1}{R^3\cdot(C1\cdot C2\cdot C3)}}{\frac{1}{R^3\cdot(C1\cdot C2\cdot C3)}+s\cdot\frac{(C3+3\cdot C2)}{R^2\cdot(C1\cdot C2\cdot C3)}+s^2\cdot\frac{2}{R\cdot C3}+s^3}$$

$$MaxGain=G=1 \qquad w_0=1/R^3\cdot(C1\cdot C2\cdot C3)$$

For a normalized 3rd order Butterworth filter the transfer function is

$$\frac{Vo}{V1}=\frac{1}{1+s\cdot 2+s^2\cdot 2+s^3}$$

Setting R=1 forces C3=1

$$\frac{Vo}{V1}=\frac{\frac{1}{(C1\cdot C2)}}{\frac{1}{(C1\cdot C2)}+s\cdot\frac{(1+3\cdot C2)}{(C1\cdot C2)}+s^2\cdot 2+s^3}=\frac{1}{1+s\cdot 2+s^2\cdot 2+s^3}$$

Then $C1\cdot C2=1$, requiring C2=1/3. The final value is then C1=3.

Summarizing: R=1, C1=3, C2=1/3, and C3=1.

3.3.4 High Q (Low Bandwidth) Bandpass Filters.

For a second-order band-pass filter the transfer function is given by

$$H(s) = \frac{V_o}{V_i} = \frac{H_o \beta s}{s^2 + \beta s + \omega_o^2}$$

where ω_o is the center frequency, β is the bandwidth and H_o is the maximum amplitude of the filter. These quantities are shown on the diagram below. The quantities in parentheses are in radian frequencies, the other quantities are in Hertz (i.e. $f_o=\omega_o/2\pi$, $B=\beta/2\pi$). Looking at the equation above, or the figure, you can see that as $\omega \to 0$ and $\omega \to \infty$ that $|H(s=j\omega)| \to 0$. You can also easily show that at $\omega=\omega_o$ that $|H(s=j\omega_o)| = H_0$. Often you will see the equation above written in terms of the quality factor, Q, which can be defined in terms of the bandwidth, β, and center frequency, ω_o, as $Q = \omega_o/\beta$. Thus the Q, or quality, of a filter goes up as it becomes narrower and its bandwidth decreases.

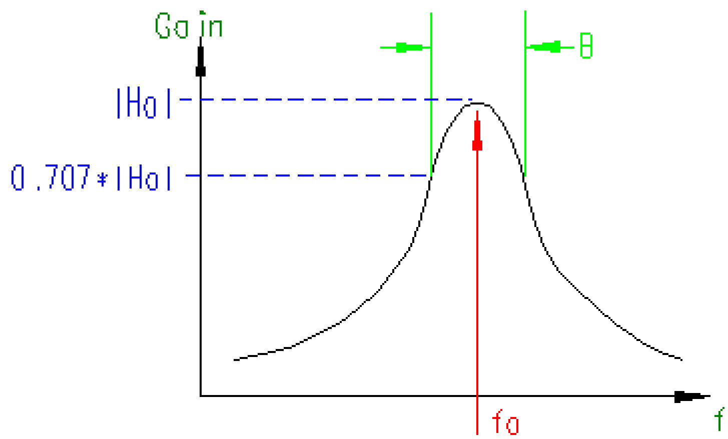


Figure 10. Defining The Q of a Filter

An active bandpass filter implementation of a unity gain Friend Circuit, also referred to as a Sallen-Key circuit as described in [7]. If you derive the transfer function of the circuit shown below:

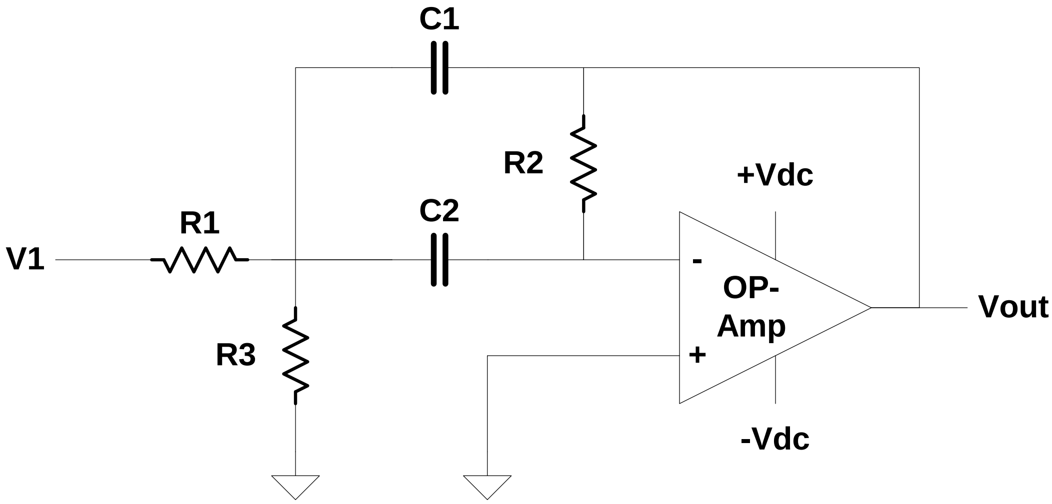


Figure 11. Friend High-Q Bandpass Filter

The transfer function for this circuit is

$$\frac{Vout}{V1} = -\frac{s \cdot \frac{1}{R_1 \cdot C_1}}{s^2 + s \cdot \frac{2}{R_2 \cdot C} + \frac{R_1 + R_3}{C^2 \cdot R_1 \cdot R_2 \cdot R_3}}$$

Under the following conditions $C1 = C2 = C$

$$MaxGain = -\frac{R_2}{2 \cdot R_1} \qquad w_o = \frac{1}{C\sqrt{R_2 \cdot \frac{R_1 \cdot R_3}{R_1 + R_3}}}$$

$$Q = \sqrt{\frac{R_1 + R_3}{2 \cdot R_3}} \qquad BW = \frac{w_0}{Q} = \frac{2}{R_2 \cdot C}$$

3.3.4.1 Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} + s \cdot C1 + s \cdot C2 \right) = \frac{V1}{R1} + V0 \cdot s \cdot C1$$

$$\frac{V0}{R2} + V2 \cdot s \cdot C2 = 0$$

$$\frac{V0}{R2} = -V2 \cdot s \cdot C2 = -s \cdot C2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} + s \cdot C1 + s \cdot C2\right)^{-1} \cdot \left(\frac{V1}{R1} + V0 \cdot s \cdot C1\right)$$

$$V0 = -\left(\frac{s \cdot R1 \cdot R2 \cdot R3 \cdot C2}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2}\right) \cdot \left(\frac{V1}{R1} + V0 \cdot s \cdot C1\right)$$

$$V0 \cdot \left(1 + \frac{s^2 \cdot R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2}\right) =$$
$$-V1 \cdot \left(\frac{s \cdot R2 \cdot R3 \cdot C2}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2}\right)$$

$$V0 \cdot \left(R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2 + s^2 \cdot R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2\right) =$$
$$-V1 \cdot \left(s \cdot R2 \cdot R3 \cdot C2\right)$$

$$V0 = -V1 \cdot \frac{s \cdot R2 \cdot R3 \cdot C2}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2 + s^2 \cdot R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}$$

$$\frac{V0}{V1} = -\frac{s \cdot R2 \cdot R3 \cdot C2}{R1 + R3 + s \cdot R1 \cdot R3 \cdot C1 + s \cdot R1 \cdot R3 \cdot C2 + s^2 \cdot R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}$$

$$\frac{V0}{V1} = -\frac{s \cdot \frac{1}{R1 \cdot C1}}{s^2 + s \cdot \left(\frac{1}{R2 \cdot C2} + \frac{1}{R2 \cdot C1}\right) + \frac{R1 + R3}{R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}}$$

Letting $C1 = C2 = C$

$$\frac{V0}{V1} = -\frac{s \cdot \frac{1}{R1 \cdot C}}{s^2 + s \cdot \frac{2}{R2 \cdot C} + \frac{R1 + R3}{R1 \cdot R2 \cdot R3 \cdot C^2}}$$

$BW = \frac{w_0}{Q} = \frac{2}{R2 \cdot C}$ and $w_0{}^2 = \frac{1}{C} \cdot \sqrt{\frac{R1 + R3}{R1 \cdot R2 \cdot R3}}$ and $Gain = \frac{R2}{2 \cdot R1}$

Letting $R2 = 2 \cdot R1$

$BW = \frac{w_0}{Q} = \frac{1}{R1 \cdot C}$ and $w_0 = \frac{1}{R1 \cdot C} \cdot \sqrt{\frac{R1 + R3}{2 \cdot R3}}$

3.4 Other Useful CIrcuits

3.4.1 Parallel combination of filters

Summing amplifier:

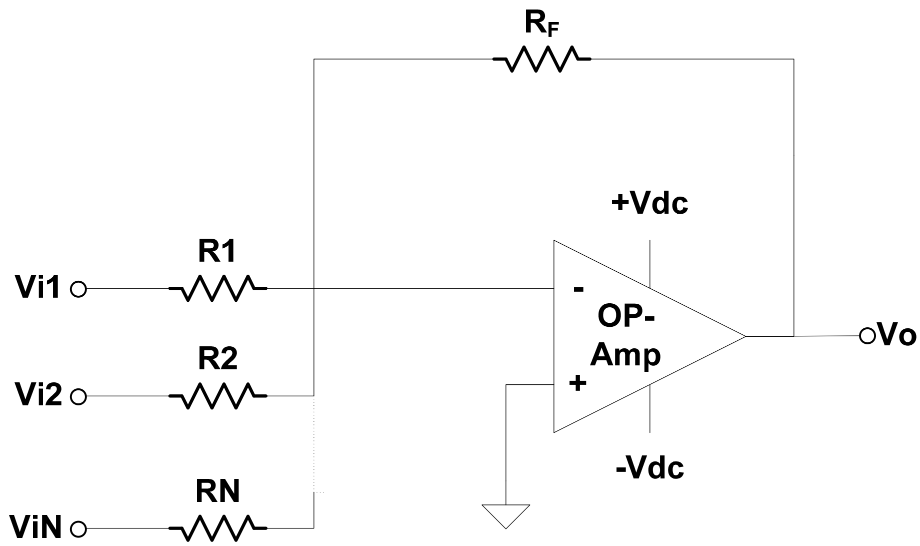


Figure 12. Op-Amp Summing Amplifier

$$V_o = -\left(\frac{R_f}{R_1} V_{i1} + \frac{R_f}{R_2} V_{i2} + \ldots\ldots + \frac{R_f}{R_N} V_{iN} \right)$$

3.4.2 Differential Amplifier:

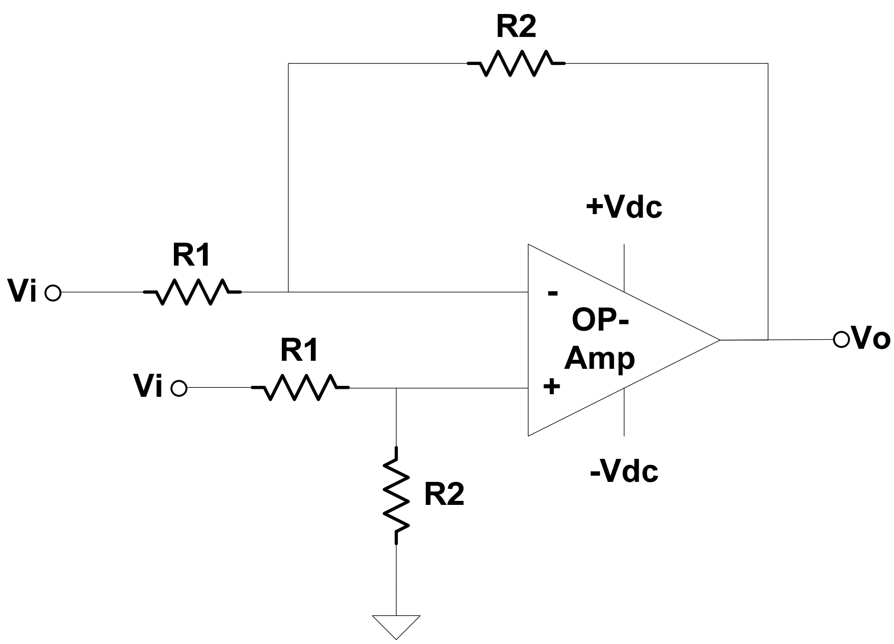


Figure 13. Op-Amp Difference or Subtraction Amplifier

$$V_o = \frac{R_2}{R_1}(V_{i2} - V_{i1})$$

4 Using Transfer Functions and OpAmp Filters to Design a Practical Filter

Take multiple stages and cascade them!

Remember to determine the pole locations that each stage of the filter requires.

As a rule-of-thumb, you should select the order for the stages of your filter. If you look at the output of each stage, it will be the product of the transfer functions to that location! So, if possible use those with damping factors closest to one before the smaller ones ….

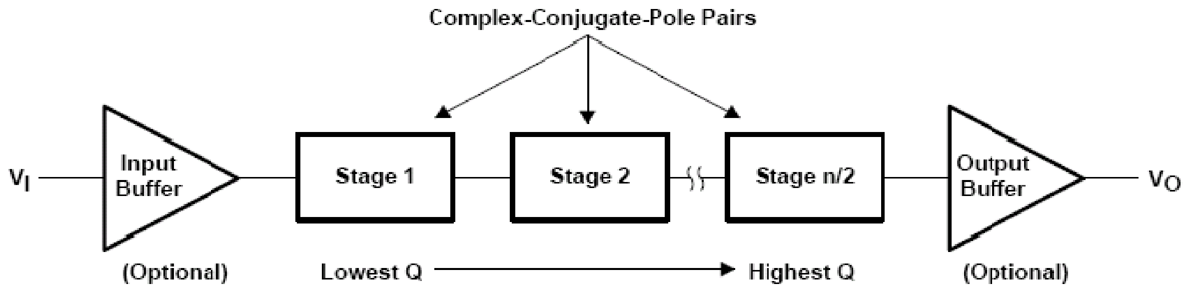


Figure 14. Building Even-Order filters by cascading second-order stages [12]
Jim Karki,Texas Instruments,Active Low-Pass Filter Design, Application Report, SLOA049B, September 2002.

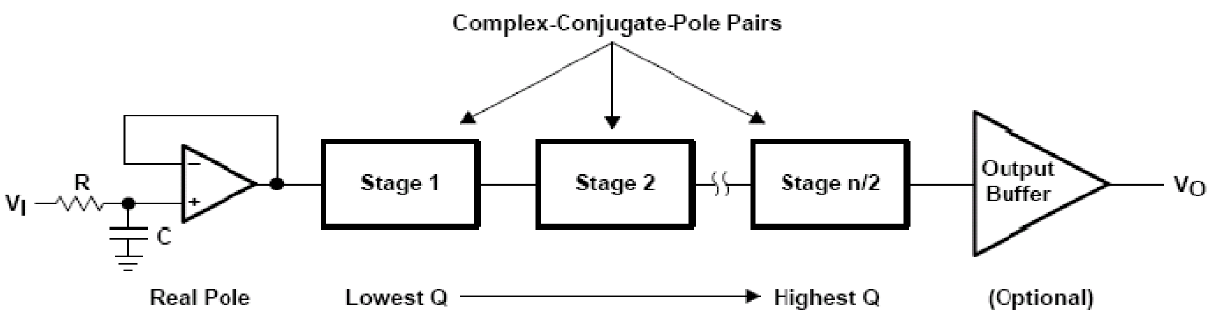


Figure 15. Building Odd-order filters by cascading second-order stages and adding a single real pole [12]
Jim Karki,Texas Instruments, Active Low-Pass Filter Design, Application Report, SLOA049B, September 2002.

Note:

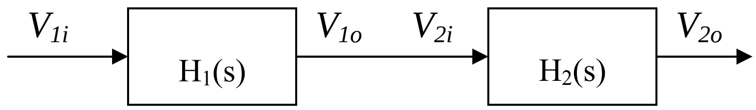
1. Real elements may not exactly match the values you select.
2. Components have a tolerance; they are within +/- some %!
3. If possible use cheaper components and one (or two) that are adjustable (potentiometers).

Cascading Filter Stages:

Given two active op-amp filter circuits with transfer functions $H_1(s)$ and $H_2(s)$. By definition, the op-amp circuit has a large input and small output impedance.

In the limit that the input and output impedances tend to infinity and zero, then cascading the filters yields a transfer function equal to the product of the individual circuit transfer functions:

$$H_T(s) = H_1(s)H_2(s)$$



$$H_T(s) = \left(\frac{V_{2o}}{V_{1i}}\right) = \left(\frac{V_{1o}}{V_{1i}}\right)\left(\frac{V_{2o}}{V_{2i}}\right) = H_1(s)H_2(s)$$

(If the output impedance of H_1, or the input impedance of H_2 are not large, this relationship is no longer true)

Example: Derive the transfer function for a first-order low-pass filter (with $A_v = 2$) and fc = 1 kHz) in series with a second order band-reject filter (with $f_0 = 500$ Hz, B = 200 Hz, and A_v=1).

Solution:

$$H_{LP}(s) = \frac{A_v}{1 + \frac{s}{\omega_c}} = \frac{2}{1 + \frac{s}{2\pi \cdot 1 \times 10^3}}$$

$$H_{BR}(s) = \frac{\left(s^2 + \omega_o^2\right)A_v}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} = \frac{\left(s^2 + (2\pi \cdot 500)^2\right)\cdot 1}{s^2 + 2\pi \cdot 200s + (2\pi \cdot 500)^2}$$

5 Component References

E12 Component Values (capacitor, inductors and resistors)

1.0	1.2	1.5	1.8	2.2	2.7
3.3	3.9	4.7	5.6	6.8	8.2

and all multiples of 10. These are common 10% tolerance resistor values.

E24 Component Values (capacitor, indiuctors, and resistors)

1.0	1.1	1.2	1.3	1.5	1.6
1.8	2.0	2.2	2.4	2.7	3.0
3.3	3.6	3.9	4.3	4.7	5.1
5.6	6.2	6.8	7.5	8.2	9.1

and all multiples of 10. These are common 5% tolerance resistor values.

E96 Component Values (resistors only)

1.0	1.02	1.05	1.07	1.10	1.13
1.15	1.18	1.21	1.24	1.27	1.30
1.33	1.37	1.40	1.43	1.47	1.50
1.54	1.58	1.62	1.65	1.69	1.74
1.78	1.82	1.87	1.91	1.96	2.00
2.05	2.10	2.15	2.21	2.26	2.32
2.37	2.43	2.49	2.55	2.61	2.67
2.74	2.80	2.87	2.94	3.01	3.09
3.16	3.24	3.32	3.40	3.48	3.57
3.65	3.74	3.83	3.92	4.02	4.12
4.22	4.32	4.42	4.53	4.64	4.75
4.87	4.99	5.11	5.23	5.36	5.49
5.62	5.76	5.90	6.04	6.19	6.34
6.49	6.65	6.81	6.98	7.15	7.32
7.50	7.68	7.87	8.06	8.25	8.45
8.66	8.87	9.09	9.31	9.53	9.76

and all multiples of 10. These are available 1% tolerance resistor values.

Rules of thumb for selecting component ranges from Dr. Bazuin:

Component Type	Range of Values
Signal Capacitors	10 pF to 0.1 µF
Tantalum/Electrolytic Capacitors	1 µF to 47 µF
Resistors	50Ω to 560 kΩ
Inductors	2.7 nH to 1000 µH

6 Components: Manufacturers and Sales

6.1 Manufacturers

All component manufacturers maintain web site information. Most web sites will provide general information and data sheets for their products. Some manufacturers provide sample quantities for free and/or provide for direct sales to customers.

6.1.1 Active Components

A general list of active analog and sum digital components include:

Analog Devices:	www.analog.com
National Semiconductor:	www.national.com
Linear Technologies:	www.linear.com
Texas Instruments:	www.ti.com
Maxim-Dallas Semi.:	www.maxim-ic.xom

6.1.2 Passive Components

A general list of passive components includes:

Coilcraft	http://www.coilcraft.com/
Panasonic	http://www.panasonic.com/industrial/components/
Kemet	http://www.kemet.com/

6.1.3 Component Sales

Digikey	http://www.digikey.com/
Mouser	http://www.mouser.com/
Jameco	http://www.jameco.com
Newark	http://www.newark.com/
RF Parts	http://www.rfparts.com/

7 References

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http://www.ece.cmu.edu/~ece778/lecture-notes/Motor-drive-lecture/

[2] Byter Robot for curling.
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[4] Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapoli, IN, 1974, p. 182-183.

[5] Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapolis, IN, 1974, p. 240-241.

[6] M.E. Van Valkenburg, Analog Filter Design, Oxford, 1982. ISBN: 0-19-510734-9.

[7] Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapolis, IN, 1974, p. 499-500.

[8] Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapolis, IN, 1974, p. 500-501.

[9] http://www.circuitsage.com/filter.html

[10] http://focus.ti.com/analog/docs/techdocs.tsp?contentType=8&familyId=78&navSection=app_notes

[11] Improved roll-off of Sallen-Key Filter, EDN September 30, 2004, p. 88. www.edn.com

[12] Jim Karki,Texas Instruments, Active Low-Pass Filter Design, Application Report, SLOA049B, September 2002.
http://focus.ti.com/general/docs/litabsmultiplefilelist.tsp?literatureNumber=sloa049b

TI Application Notes:
- Slod006b
- Sloa093

TI Application Notes on Filtering
- Active Filter Design Techniques SLOA088
- Analysis of the Sallen-Key Architecture (Rev. B) SLOA024
- FilterPro MFB and Sallen-Key Low-Pass Filter Design Program SBFA001A
- Using the Texas Instruments Filter Design Database SLOA062
- Filter Design in Thirty Seconds SLOA093
- Filter Design on a Budget SLOA065
- More Filter Design on a Budget SLOA096