The turnpike property: a classical feature of optimal control problems revisited

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based on joint work with Roberto Guglielmi (Waterloo),
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Western Michigan University, Kalamazoo
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Outline

- The turnpike property
- Relation to other system theoretic properties
- Application: Model Predictive Control

System class

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x$$

with $x_{\mathbf{u}}(n) \in X$, $\mathbf{u}(n) \in U$, $\quad X$, U normed spaces

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Brief notation $x^+ = f(x,u)$

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$$\operatorname*{minimise}_{\mathbf{u}} \; J_N(x,\mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(n),\mathbf{u}(n))$$

with a cost function $\ell : X \times U \to \mathbb{R}$ and state and input constraints $x_{\mathbf{u}}(n) \in \mathbb{X}$, $\mathbf{u}(n) \in \mathbb{U}$

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We illustrate the property by two simple examples

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Example 1: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x^+ = 2x + u$$

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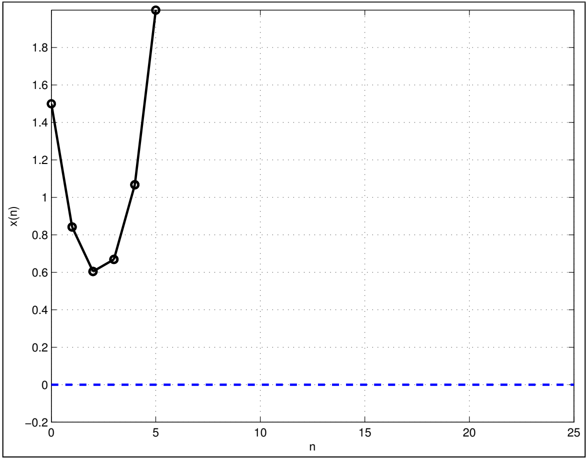
$\rightsquigarrow$ optimal trajectory should stay near $x^e = 0$

Example 1: optimal trajectories



Optimal trajectory for $N = 5$

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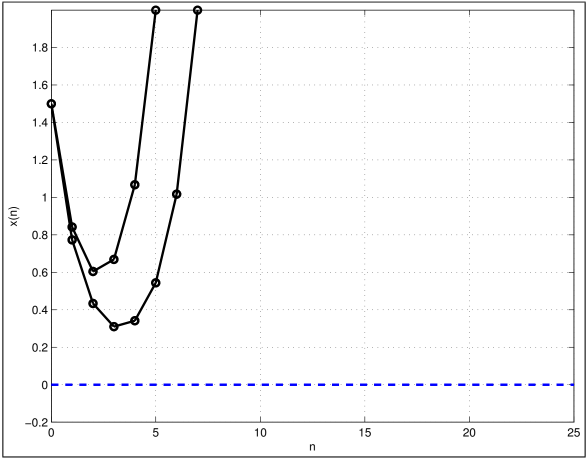

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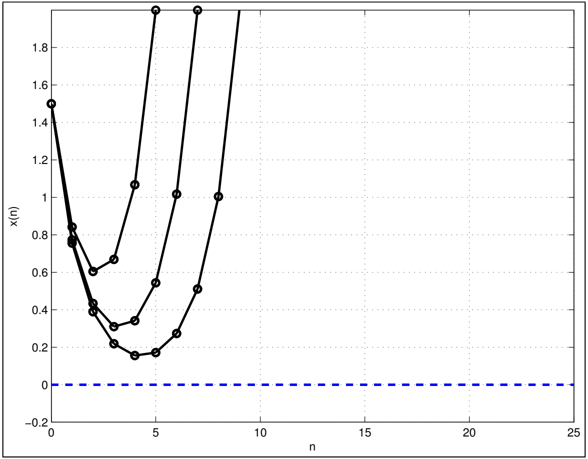

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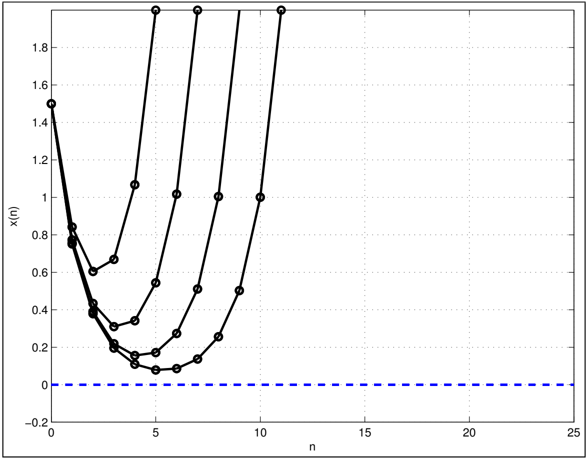

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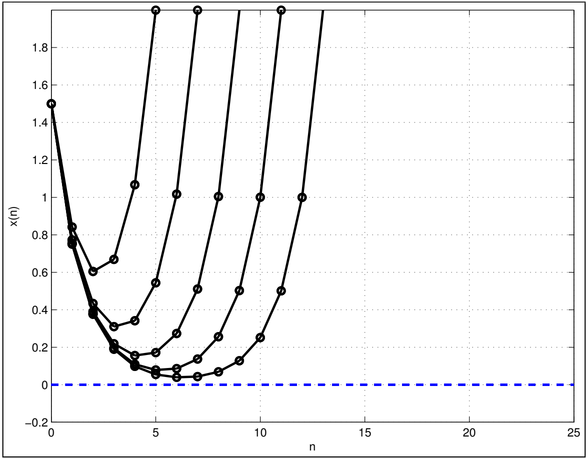

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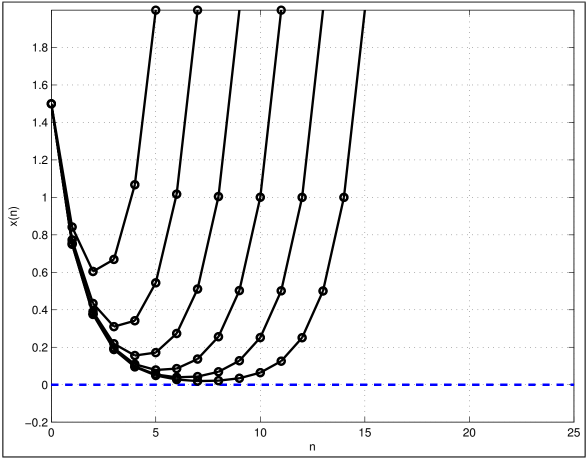

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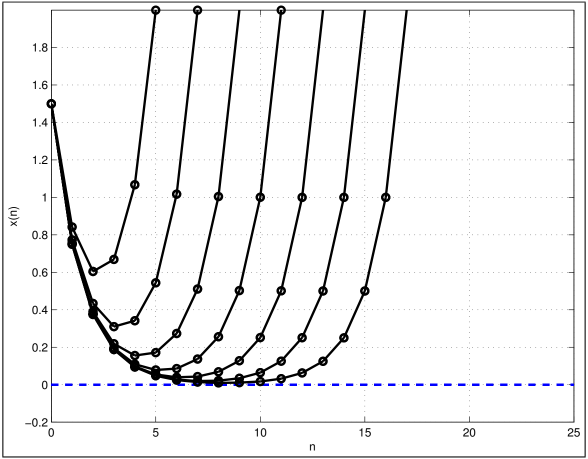

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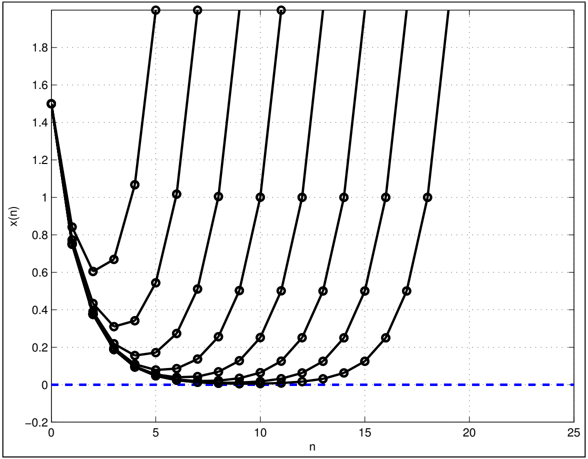

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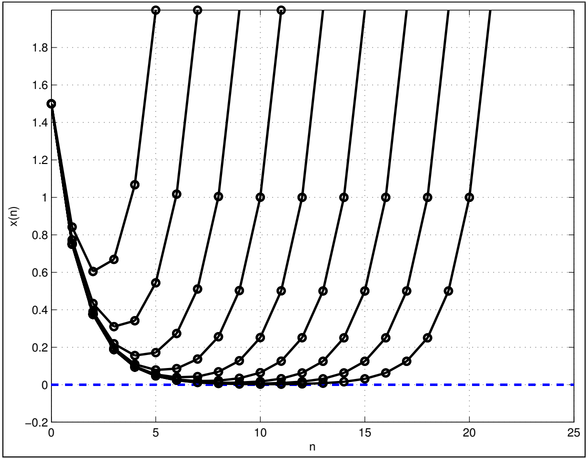

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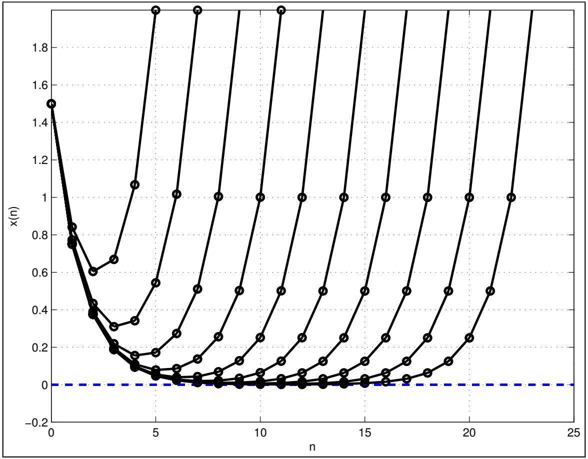

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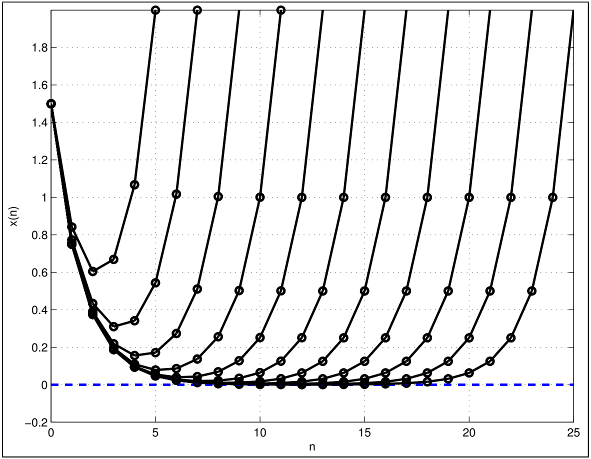

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Example 2: a macroeconomic model

Consider a classical 1d macroeconomic model

[Brock/Mirman '72]

Minimize the finite horizon objective $\sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k))$ with

$$\ell(x,u) = -\ln(Ax^{\alpha} - u), \quad A = 5,\ \alpha = 0.34$$

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On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

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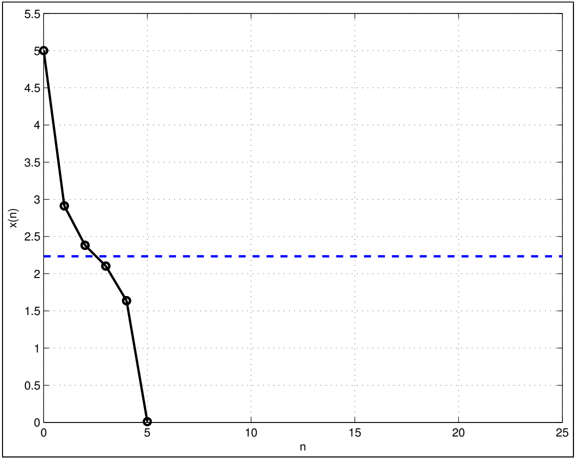

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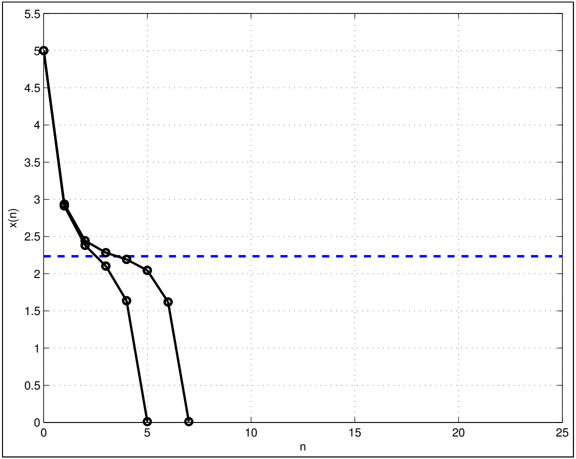

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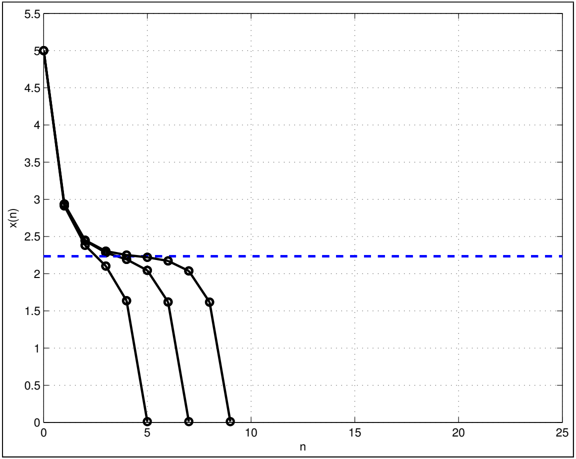

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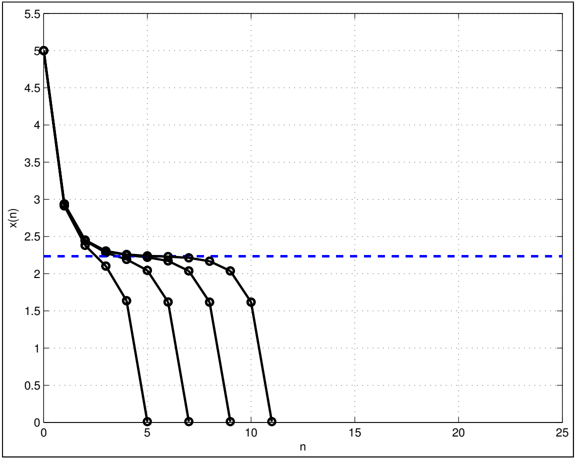

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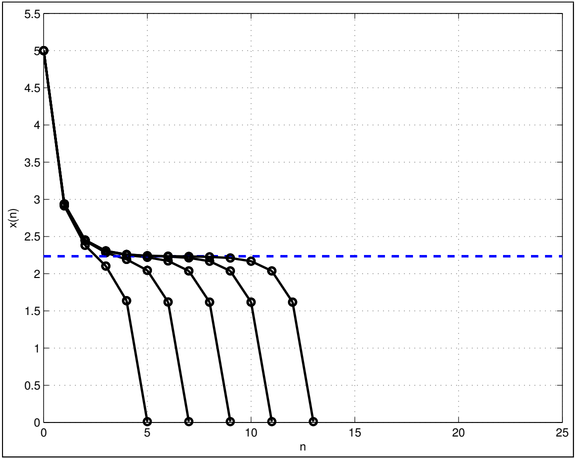

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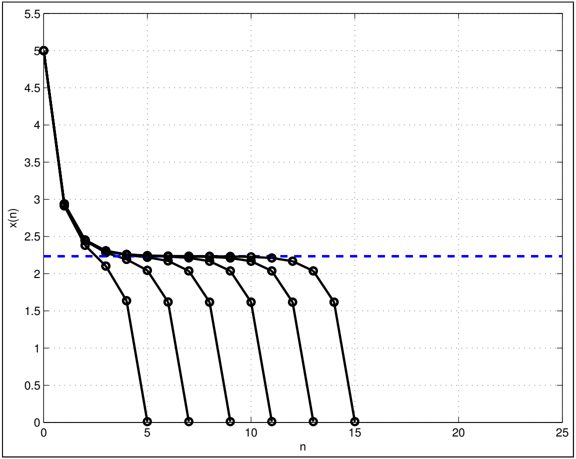

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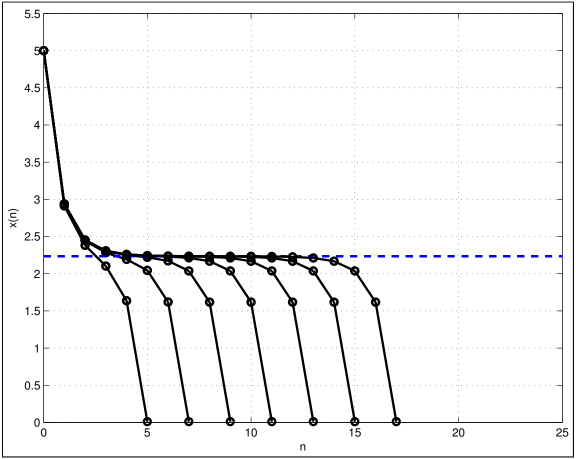

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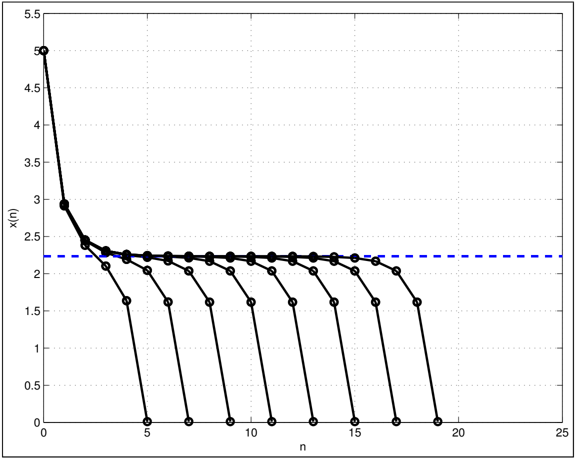

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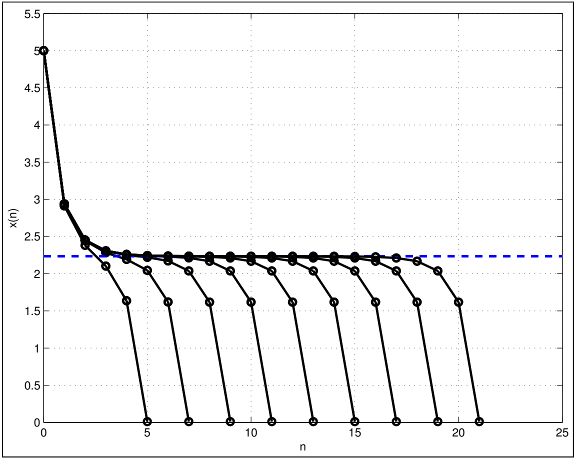

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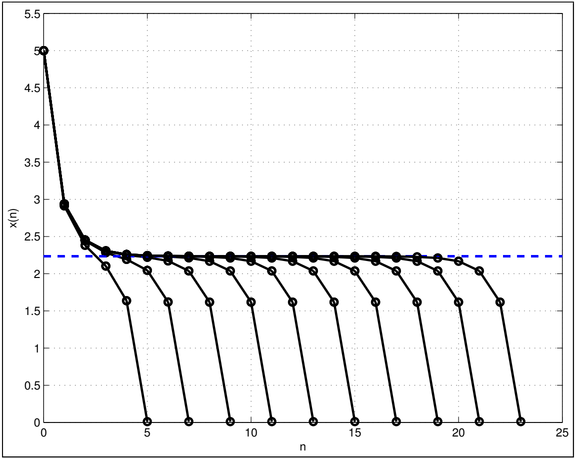

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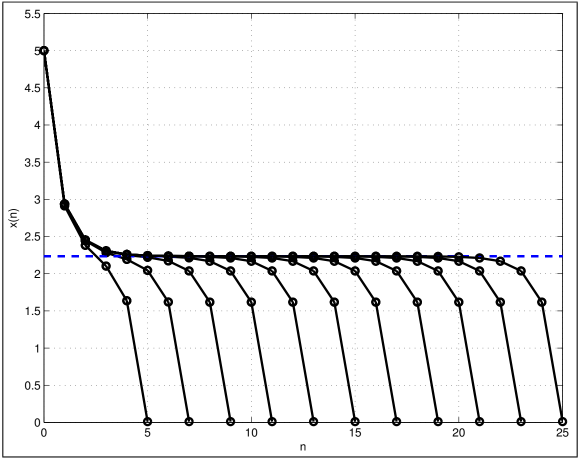

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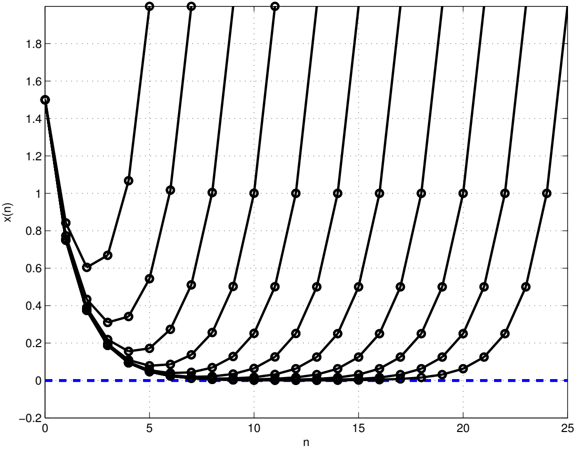

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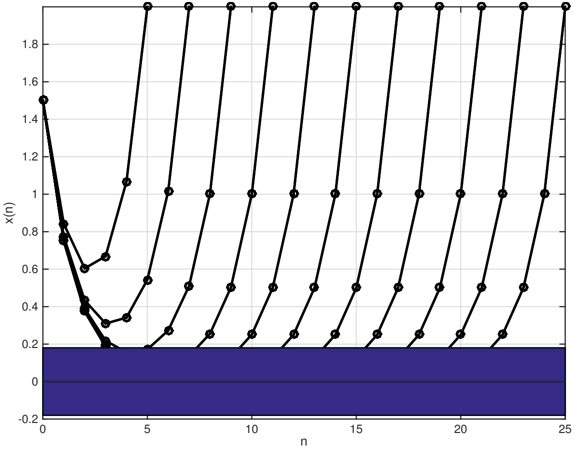

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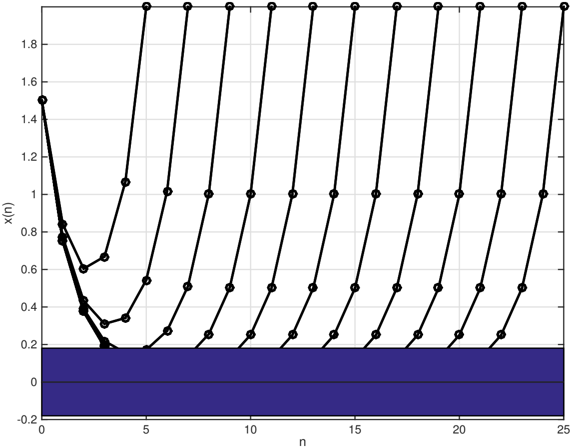

Number of points outside the blue neighbourhood is bounded by a number independent of N (here: by 8)

The turnpike property: formal definitions

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Turnpike property: For each $\varepsilon > 0$ and $\rho > 0$ there is $C_{\rho,\varepsilon} > 0$ such that for all $N \in \mathbb{N}$ all optimal trajectories $x^\star$ starting in $B_\rho(x^e)$ satisfy the inequality

$$\#\left\{k \in \{0, \dots, N-1\} \,\middle|\, \|x^\star(k) - x^e\| \geq \varepsilon\right\} \leq C_{\rho,\varepsilon}$$

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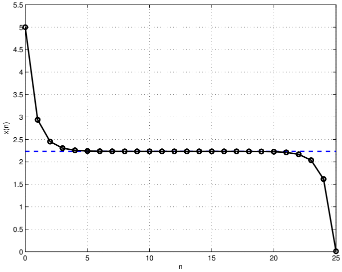

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- Closely linked to dissipativity [Willems '72]

(Strict) dissipativity and detectability

Strict dissipativity

The optimal control problem is called strictly dissipative on a set $\mathbb{X} \subset X$, if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, and $\alpha \in \mathcal{K}$ such that for all $x \in \mathbb{X}$, $u \in U$ with $f(x, u) \in \mathbb{X}$:

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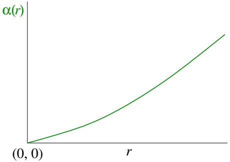
$\alpha \in \mathcal{K}$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$



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Facts (for "sufficiently fast" stabilizable systems):

- Strict dissipativity implies the turnpike property [Carlson et al. '91; Gr. '13]

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$\Rightarrow \lambda(x^\star(K)) - \lambda(x^\star(0)) + \sum_{k=0}^{K-1} \alpha(\|x^\star(k) - x^e\|) < C$

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Strict dissipativity

The optimal control problem is called strictly dissipative on a set $\mathbb{X} \subset X$, if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, and $\alpha \in \mathcal{K}$ such that for all $x \in \mathbb{X}$, $u \in U$ with $f(x,u) \in \mathbb{X}$:

$$\lambda(f(x,u)) \leq \lambda(x) + \ell(x,u) - \ell(x^e,u^e) - \alpha(\|x - x^e\|)$$

Facts (for "sufficiently fast" stabilizable systems):

- Strict dissipativity implies the turnpike property [Carlson et al. '91; Gr. '13]
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Dissipativity links to classical systems theoretic properties for linear-quadratic problems

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Strict dissipativity and detectability

Consider a linear quadratic finite dimensional discrete time problem with

$$x^+ = Ax + Bu, \quad \ell(x,u) = x^T Q x + u^T R u + s^T x + v^T u$$

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Work in progress: extension to infinite dimensional systems [Gr./Muff/Schaller '21]

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For nonlinear systems, detectability can be expressed as follows:

There are functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite and $W : \mathbb{X} \to \mathbb{R}_{\geq 0}$ continuous, such that

$$\begin{aligned} W(x) &\leq \alpha_1(\|x\|) \\ W(f(x,u)) - W(x) &\leq -\alpha_2(\|x\|) + \gamma(\ell(x,u)) \end{aligned}$$

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Sensitivity

Setting

Now we switch to continuous time and consider general linear evolution equations

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Solution concept: mild solution

Optimal control problem

Optimality condition: Pontryagin's Maximum Principle yields

$$\underbrace{\begin{pmatrix} C^*C & -\frac{d}{dt} - A^* \\ 0 & E_T \\ \frac{d}{dt} - A & -BQ^{-1}B^* \\ E_0 & 0 \end{pmatrix}}_{=:\,M} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix},$$

where $E_0 y := y(0)$ and $E_T \lambda := \lambda(T)$

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Idea: Use $\|\delta y\| + \|\delta\lambda\| \le \|M^{-1}\| \|(\varepsilon_1, 0, \varepsilon_2, 0)\|$

plus exponential weighting

Sensitivity result

Theorem [Gr./Schaller/Schiela 20]: Define $\rho := \|e^{-\mu t}\varepsilon_1(t)\|_{L_p(X)}$ $+\|e^{-\mu t}\varepsilon_2(t)\|_{L_p(X)}$ for $p = 1$ or $p = 2$ and assume the norms

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$\rightsquigarrow$ Large errors for $t \approx T$ are exponentially damped at $t \approx 0$

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Boundedness of $\|M^{-1}\|$

How do we get a T-independent bound for the norms

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This is the hard part of the analysis

The turnpike property

A modification of our arguments shows that under the same conditions the exponential turnpike property follows

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It suffices to replace

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where y^e is the turnpike equilibrium and λ^e the corresponding adjoint variable. Then essentially the same analysis as before (with different exponential scaling) yields ...

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Theorem [Gr./Schaller/Schiela 20]: If the control system is exponentially stabilizable and detectable, then there are $\mu, c > 0$ independent of T such that $\delta y = y - y^e$, $\delta u = u - u^e$ and $\delta\lambda = \lambda - \lambda^e$ satisfy

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\left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta y \right\|_{L_2(X)} &+ \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta u \right\|_{L_2(U)} \\
+ \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta \lambda \right\|_{L_2(X)} &\leq c \left(\|y_0 - \bar{y}\| + \|\bar{\lambda}\| \right)
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The turnpike property

Theorem [Gr./Schaller/Schiela 20]: If the control system is exponentially stabilizable and detectable, then there are $\mu, c > 0$ independent of T such that $\delta y = y - y^e$, $\delta u = u - u^e$ and $\delta\lambda = \lambda - \lambda^e$ satisfy

$$\left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta y \right\|_{L_2(X)} + \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta u \right\|_{L_2(U)}$$
$$+ \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta \lambda \right\|_{L_2(X)} \leq c \left(\|y_0 - \bar{y}\| + \|\bar{\lambda}\| \right)$$

$$\|\delta y(t)\| + \|\delta\lambda(t)\| \leq c(e^{-\mu t} + e^{-\mu(T-t)}) \left(\|y_0 - \bar{y}\| + \|\bar{\lambda}\| \right)$$

Discussion

- This turnpike result generalizes many recent results in the literature [Gugat, Porretta, Trélat, Zuazua ...'13ff], as we require neither boundedness of B and C nor that A generates an analytic semigroup

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$\rightsquigarrow$ The dissipativity-based conditions for the turnpike property resemble Lyapunov function conditions for asymptotic stability, while the sensitivity-based approach is similar to spectral conditions for the linearisation

Model predictive control

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MPC approximates an optimal control problem on an infinite horizon

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_\infty(x,\mathbf{u}) = \sum_{n=0}^{\infty} \ell(x_\mathbf{u}(n), \mathbf{u}(n))$$

by the iterative solution of finite horizon problems

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_N(x,\mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k))$$

with fixed $N \in \mathbb{N}$

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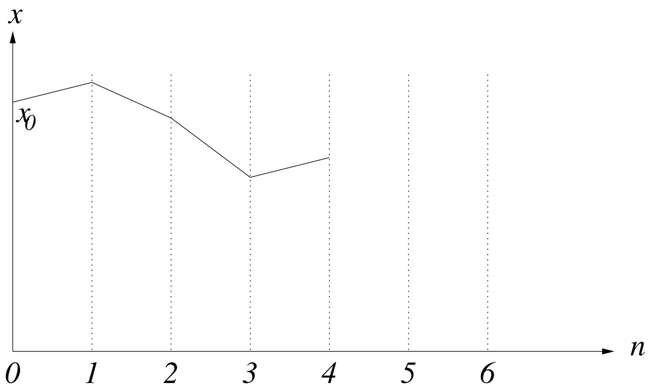
MPC from the trajectory point of view

MPC from the trajectory point of view



black = predictions (open loop optimization)

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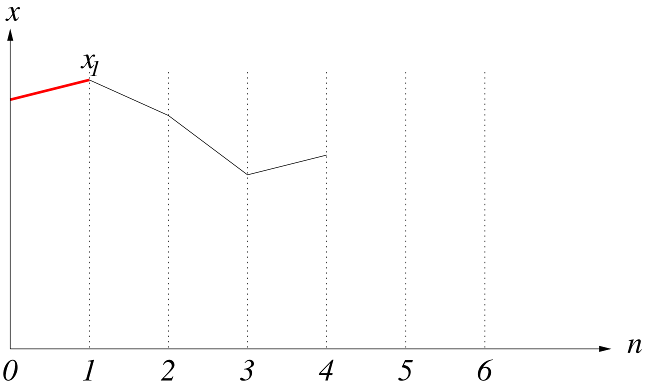

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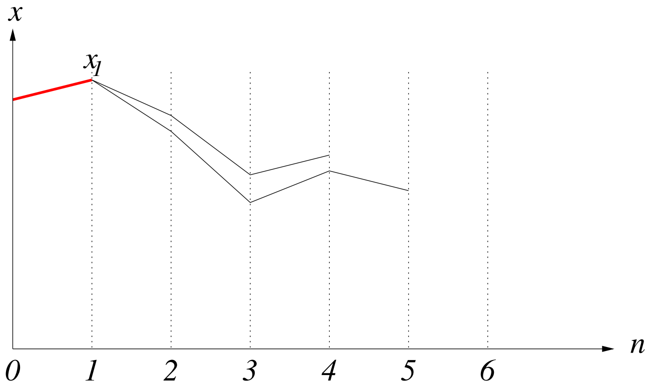

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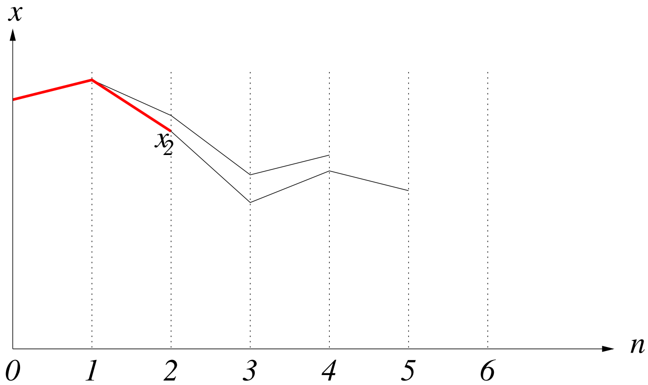

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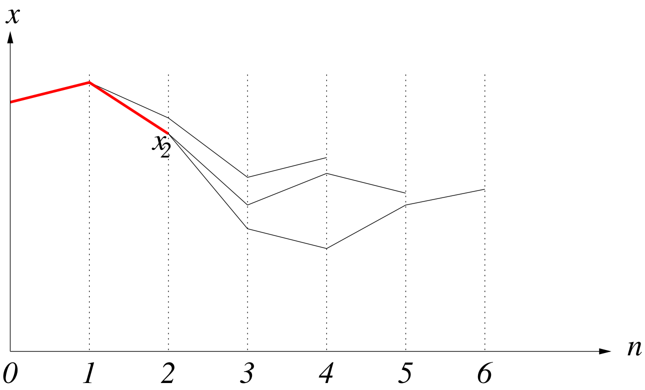

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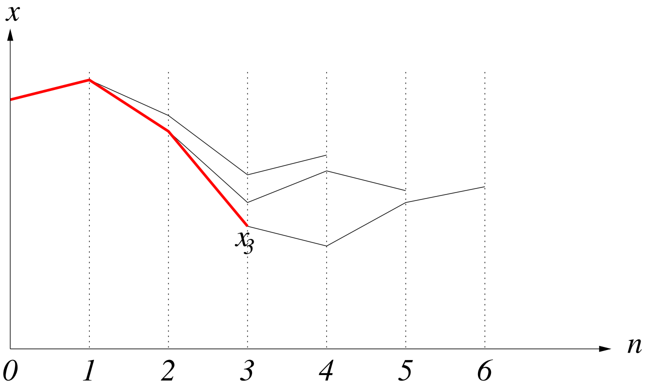

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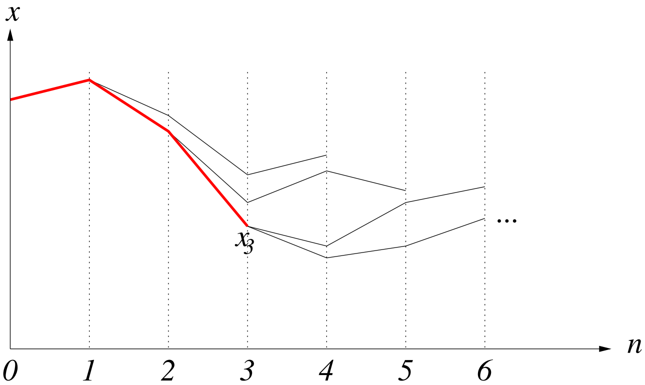

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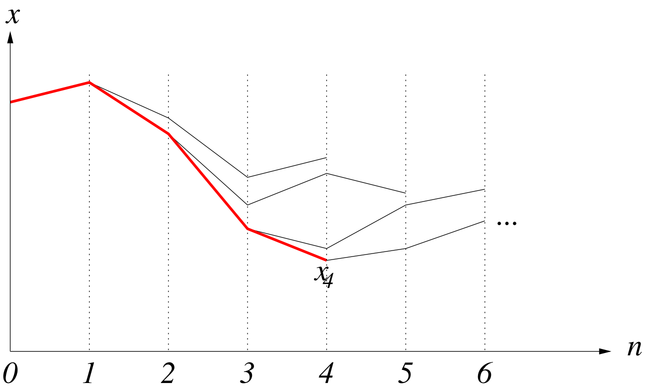

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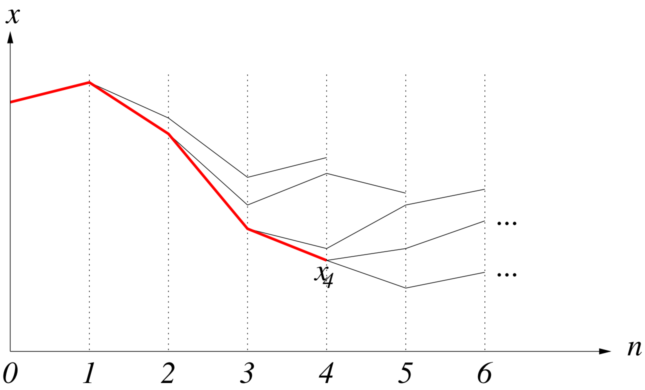

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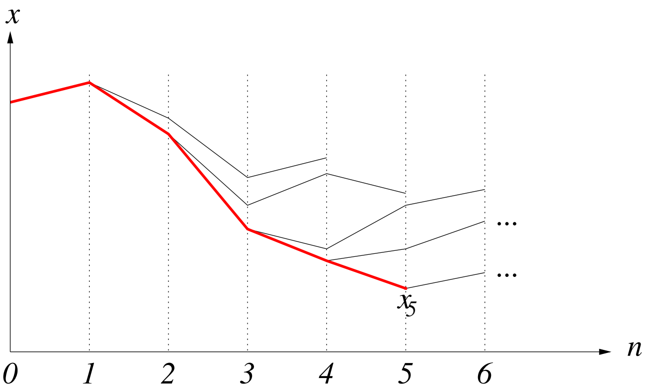

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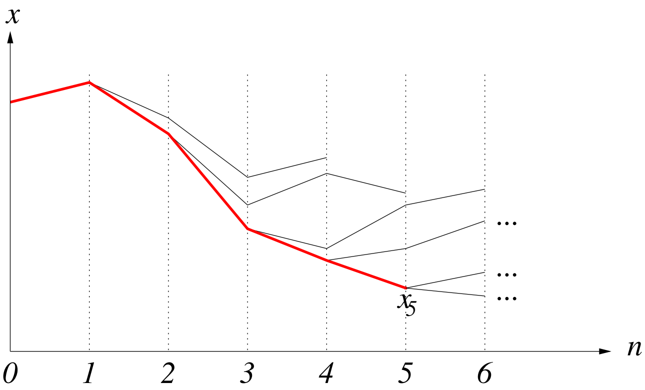

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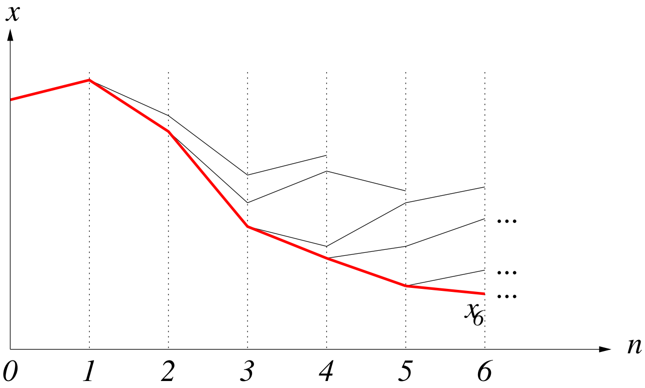

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Approximation result for MPC

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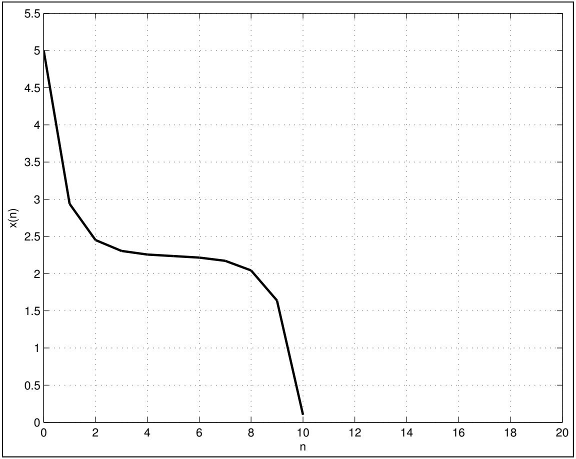
Before formalising the approximation result, we illustrate this behaviour by our second example for $N = 10$

MPC for Example 2



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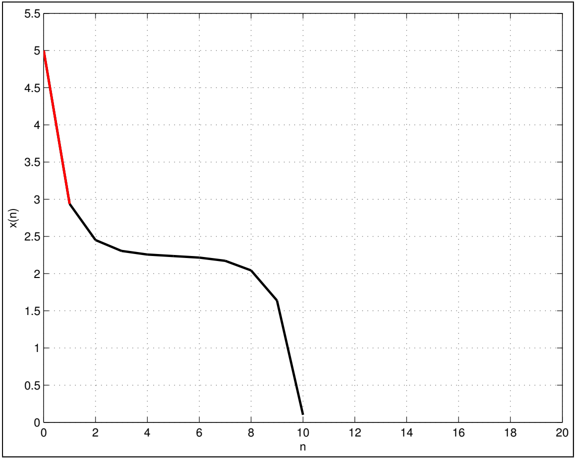

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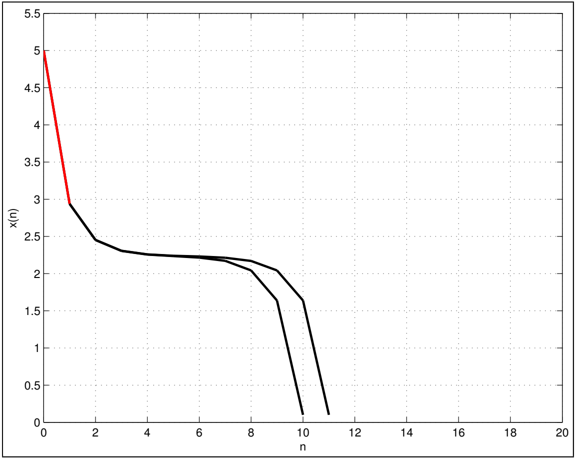

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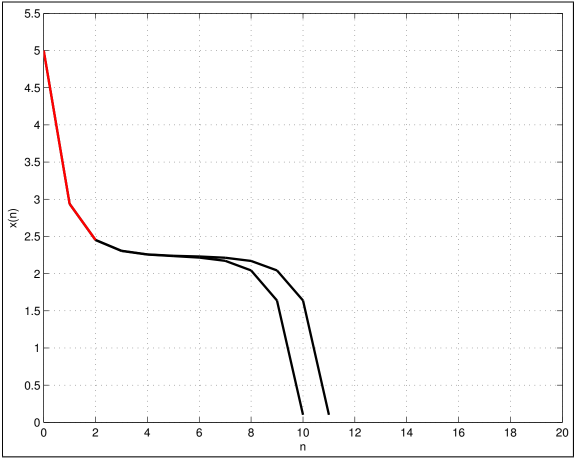

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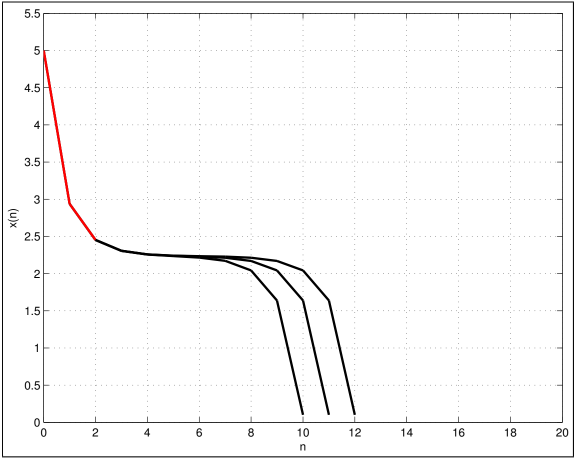

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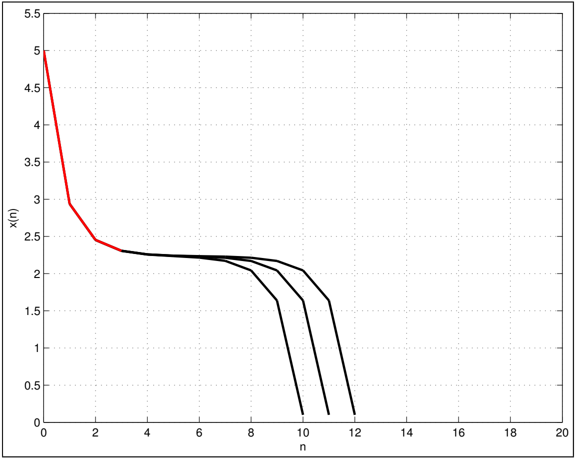

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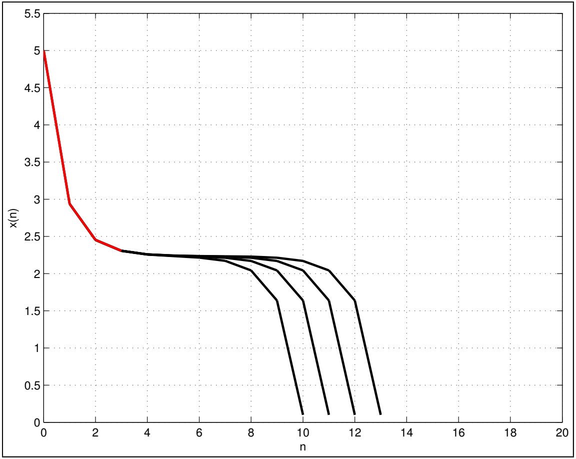

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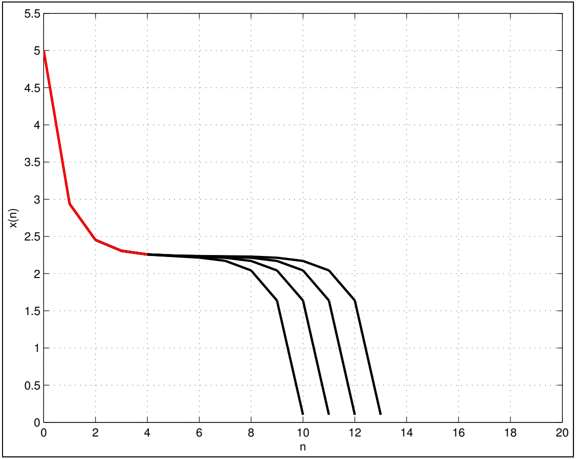

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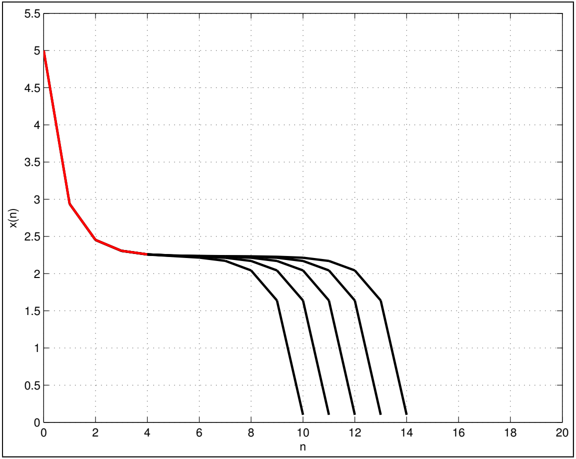

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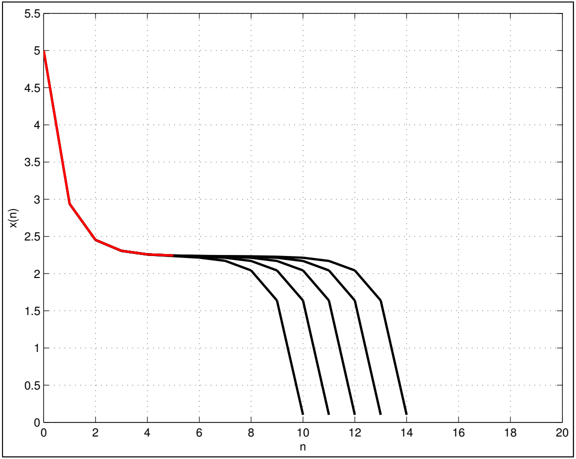

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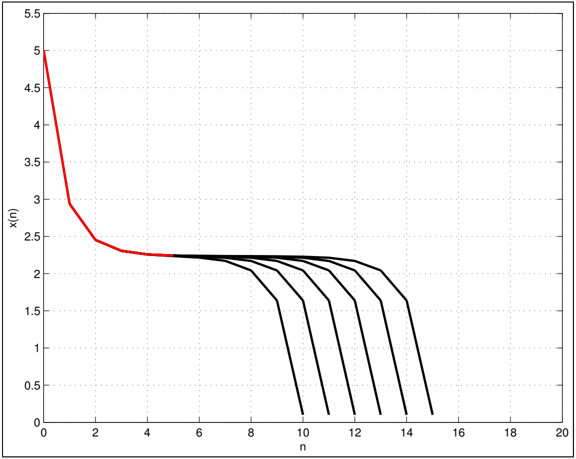

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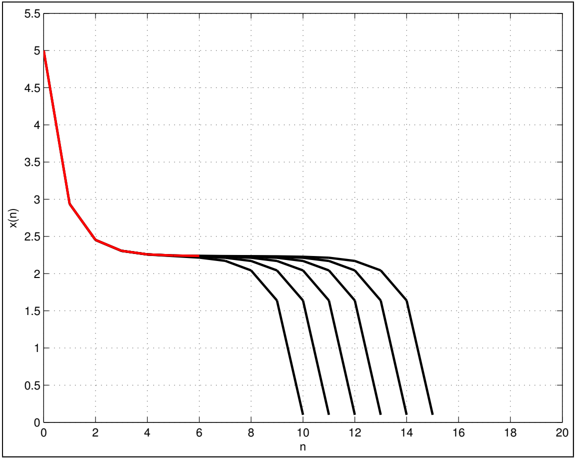

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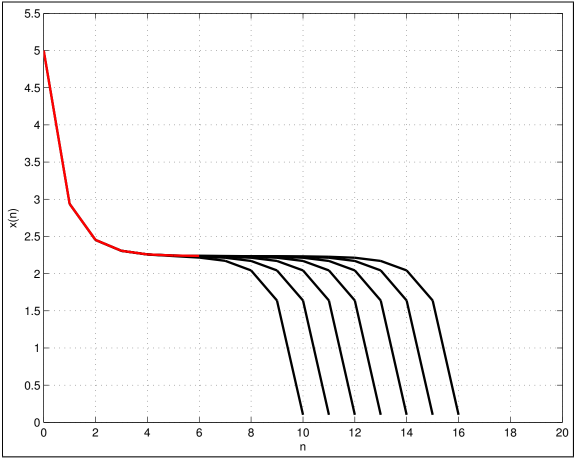

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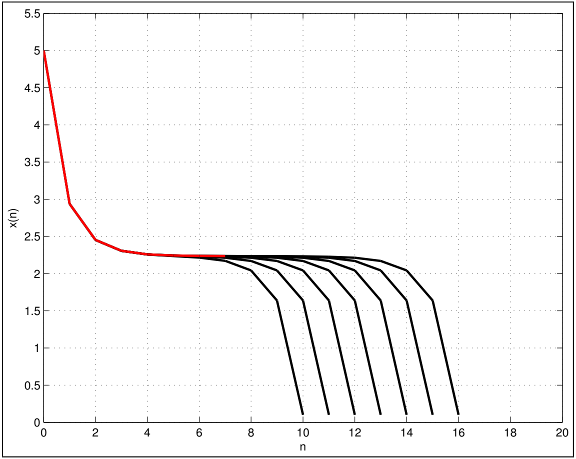

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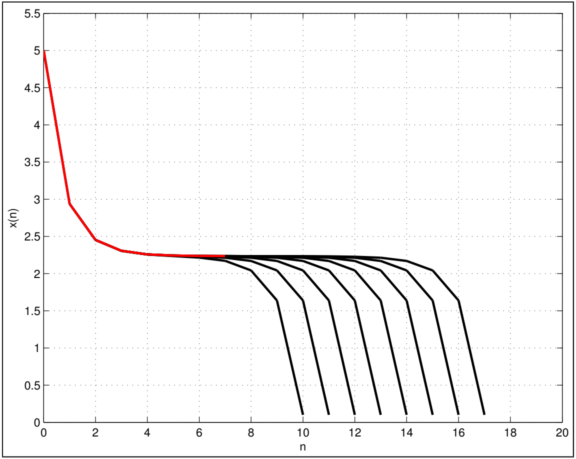

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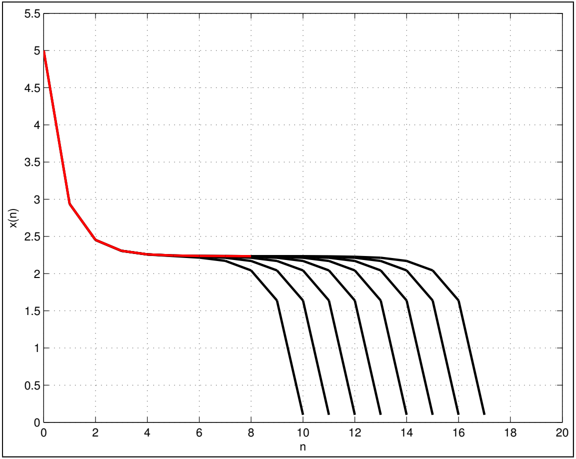

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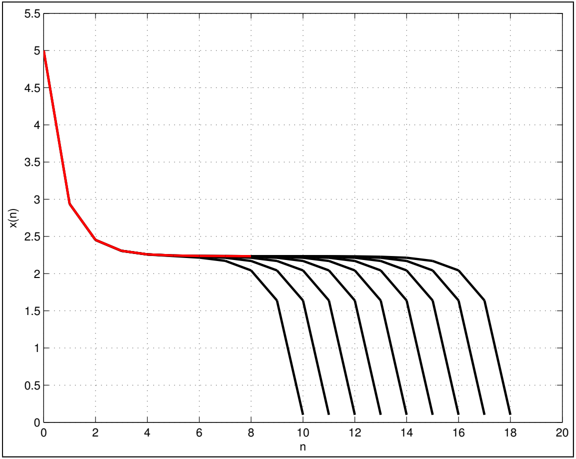

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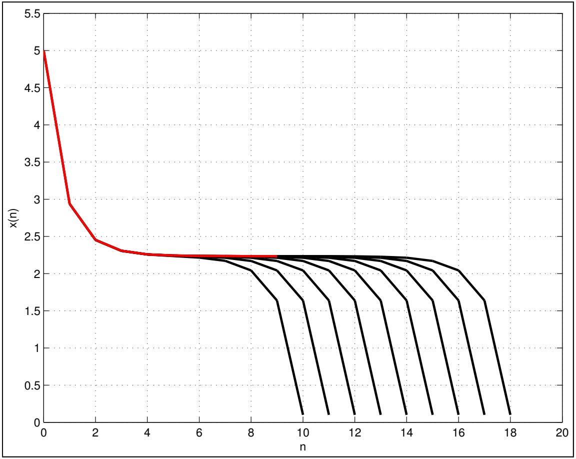

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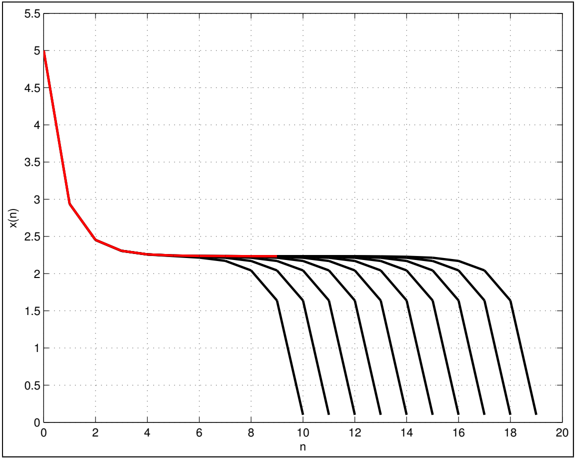

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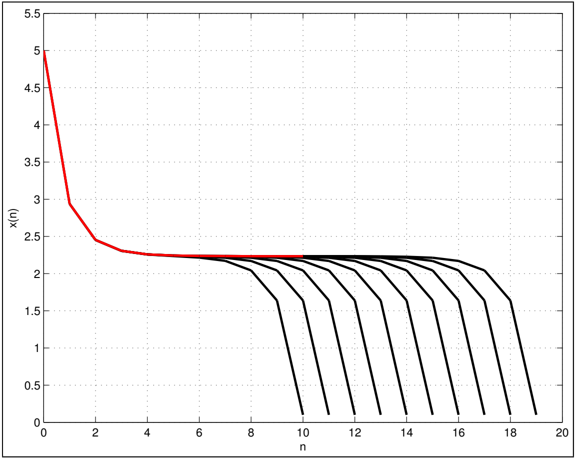

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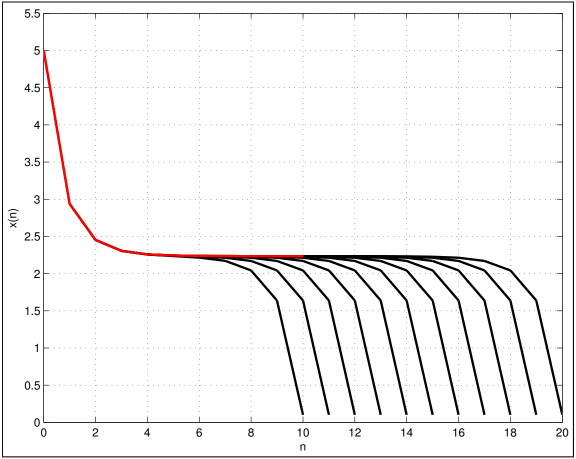

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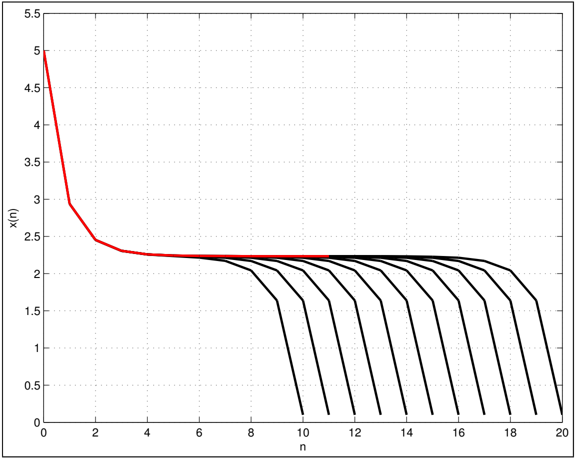

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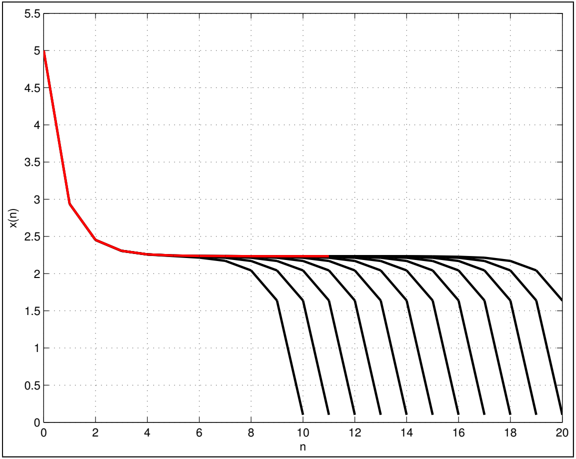

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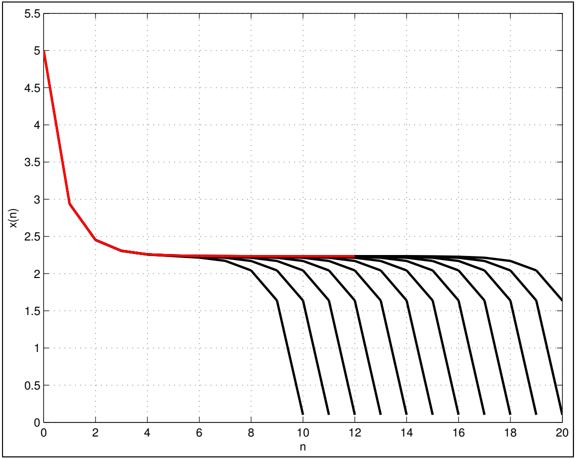

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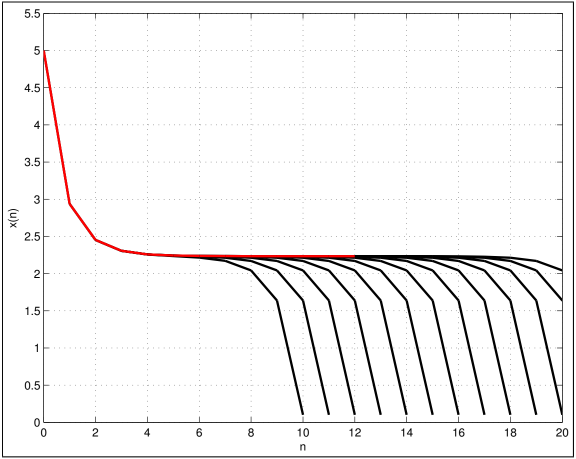

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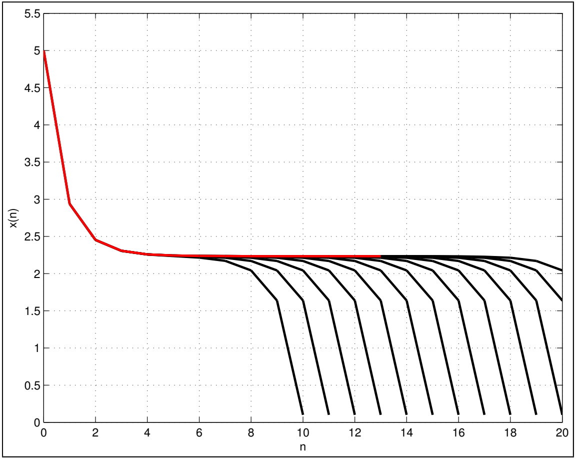

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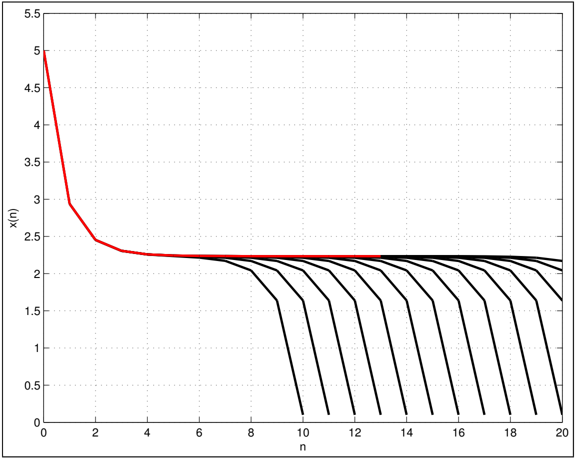

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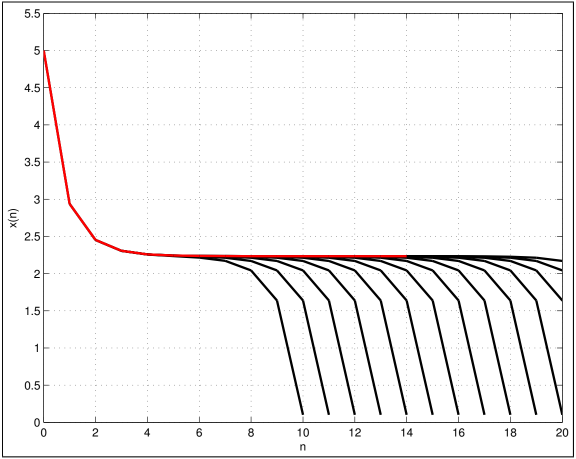

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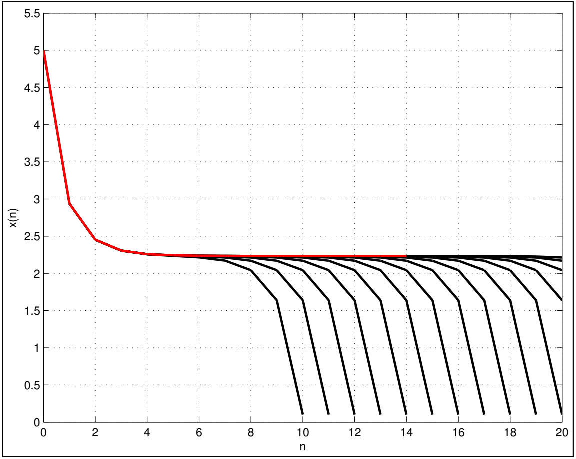

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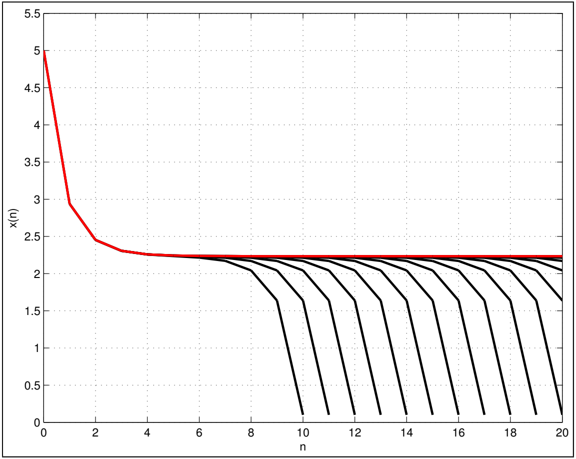

MPC via strict dissipativity — Theorem

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for all $N \in \mathbb{N} \cup \{\infty\}$ for $\gamma \in \mathcal{K}$ and $\omega(N) \to 0$ as $N \to \infty$

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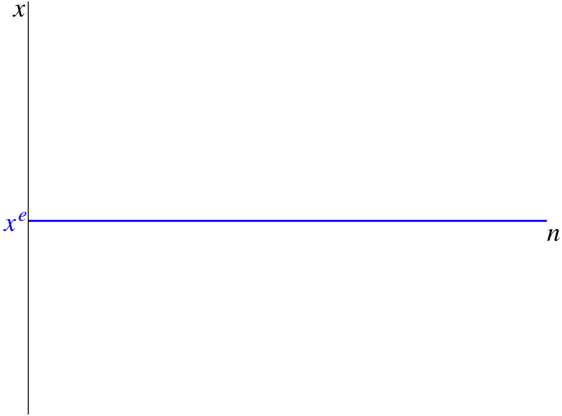
We explain the first two properties graphically

Illustration of transient optimality



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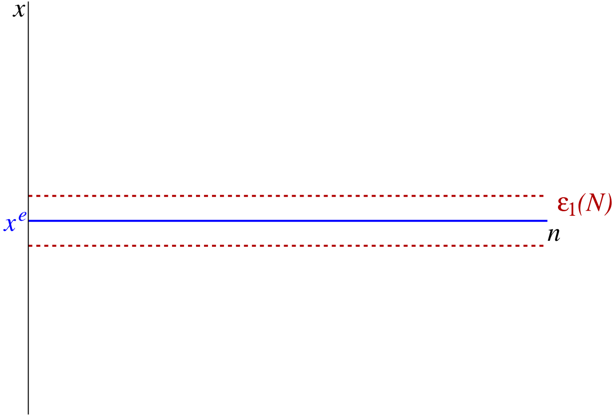

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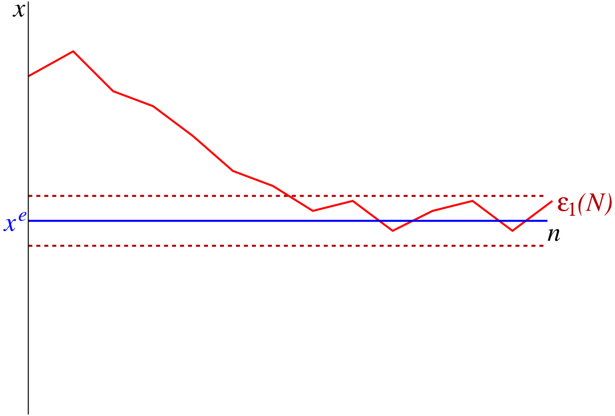

Practical asymptotic stability:

$x_{MPC}(n)$ converges to the $\varepsilon_1(N)$-ball around x^e

(with $\varepsilon_1(N) \to 0$ as $N \to \infty$)

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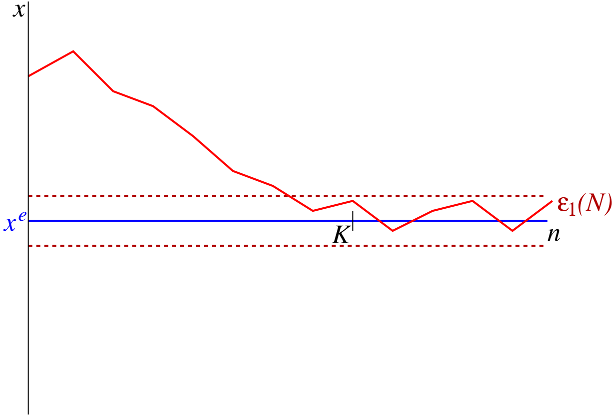

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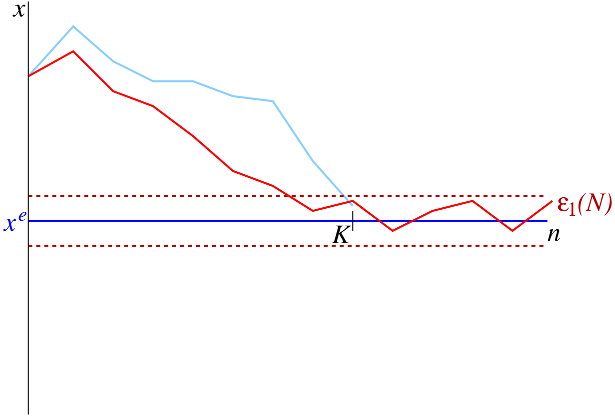

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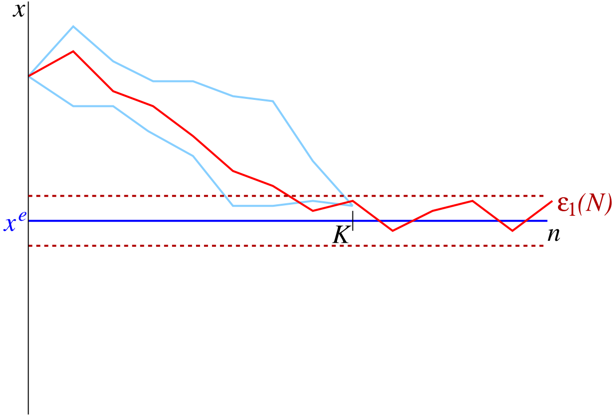

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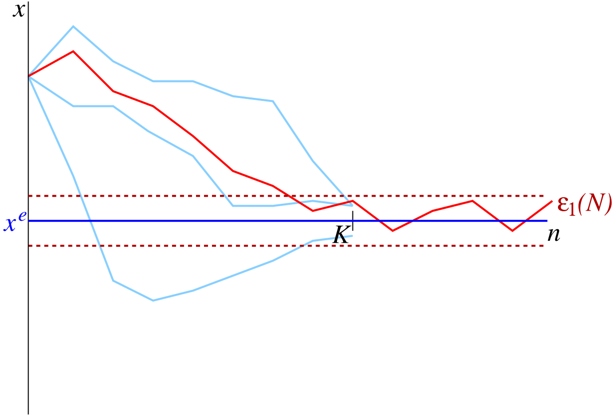

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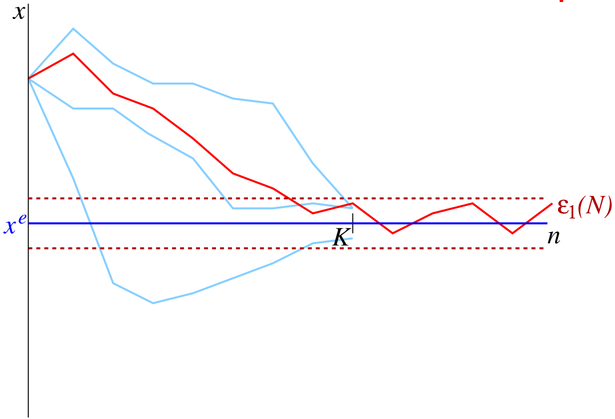

Transient optimality:

cost of all other trajectories reaching the ball at time K is higher than that of $x_{MPC}(n)$ up to an error $K\varepsilon_1(N) + \varepsilon_2(K)$ (with $\varepsilon_1(N), \varepsilon_2(K) \to 0$ as $N, K \to \infty$)

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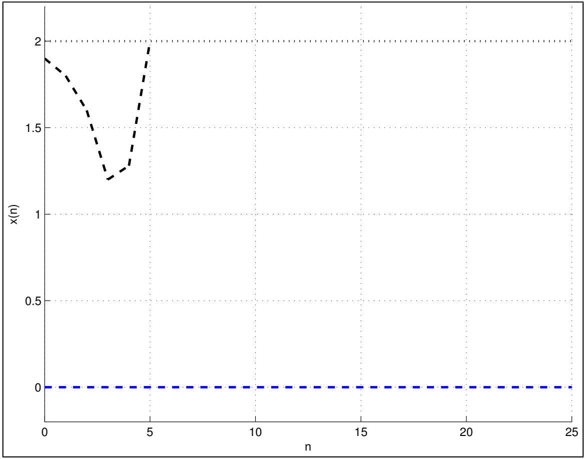
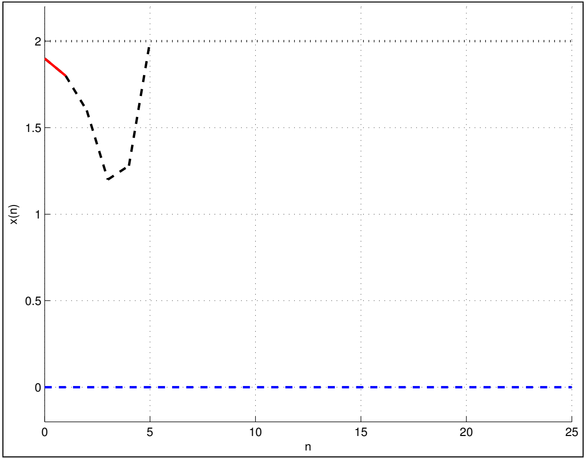
Practical asymptotic stability in Example 1



$N = 5$

Practical asymptotic stability in Example 1



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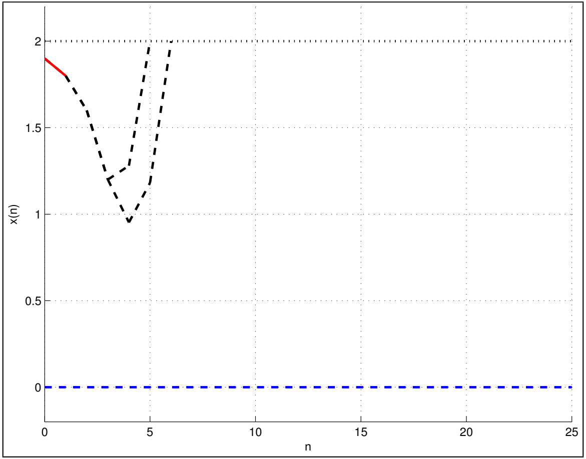

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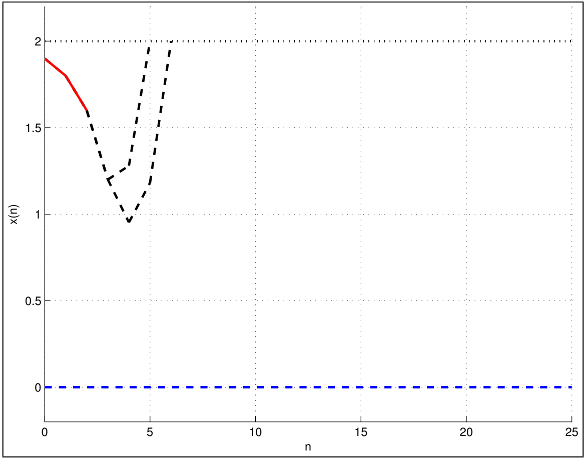

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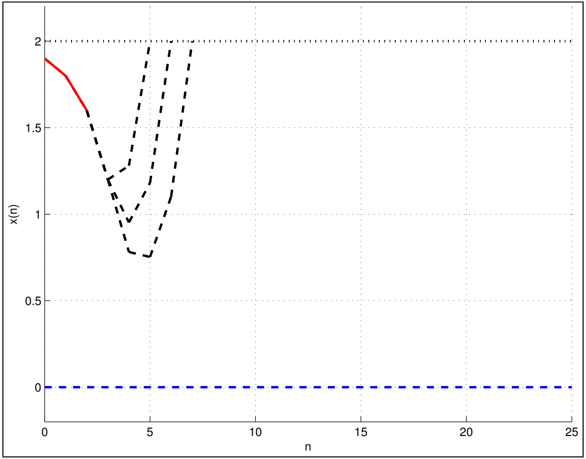

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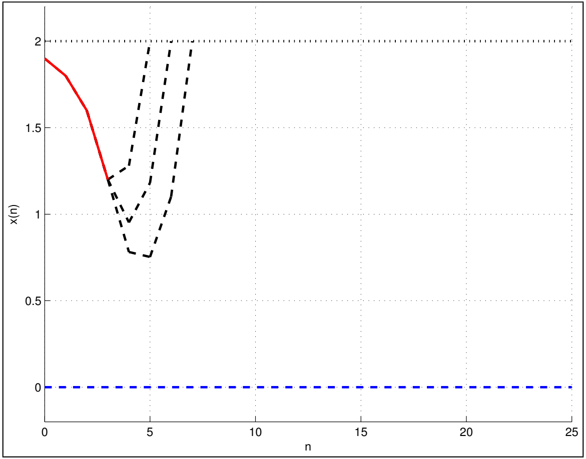

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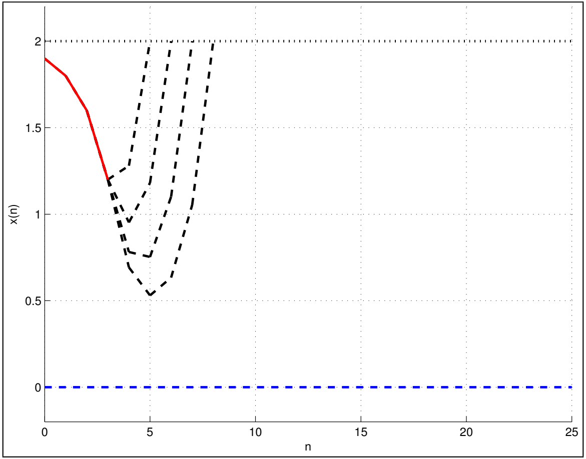

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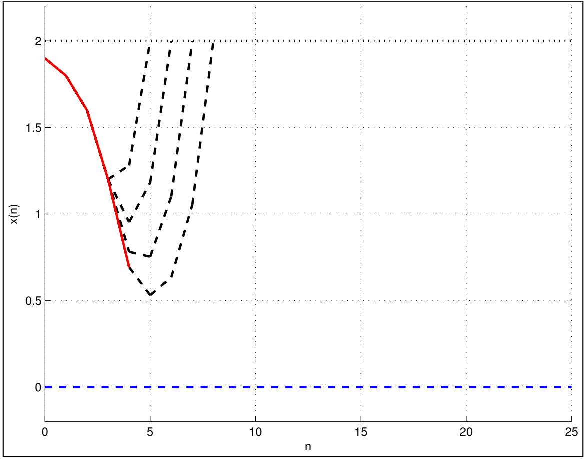

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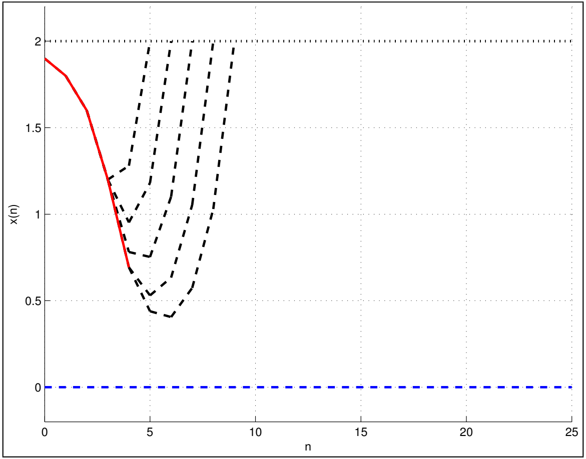

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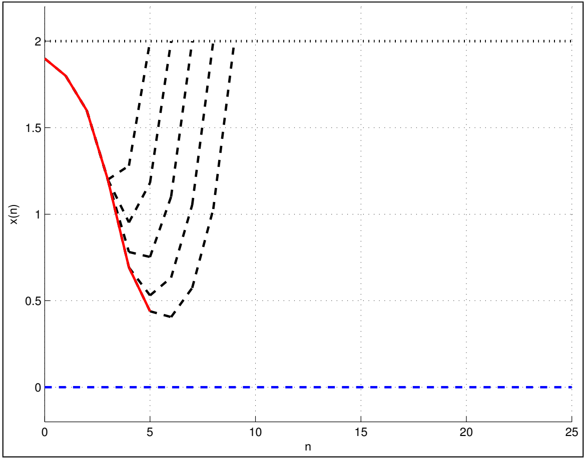

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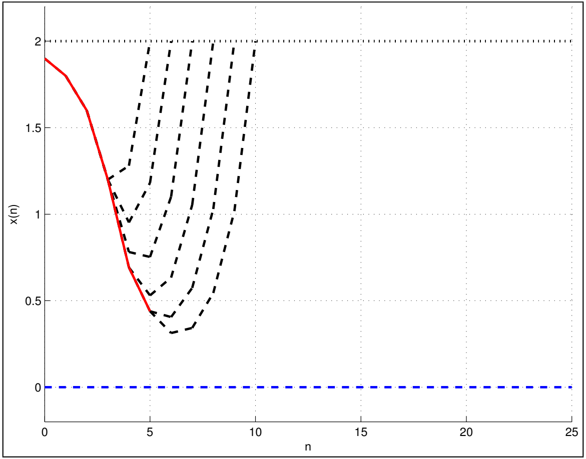

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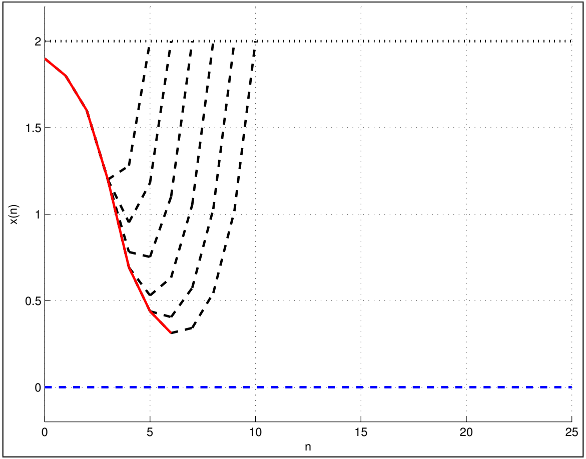

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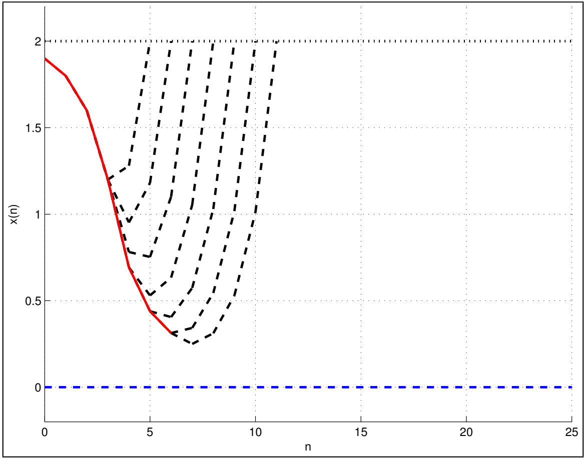

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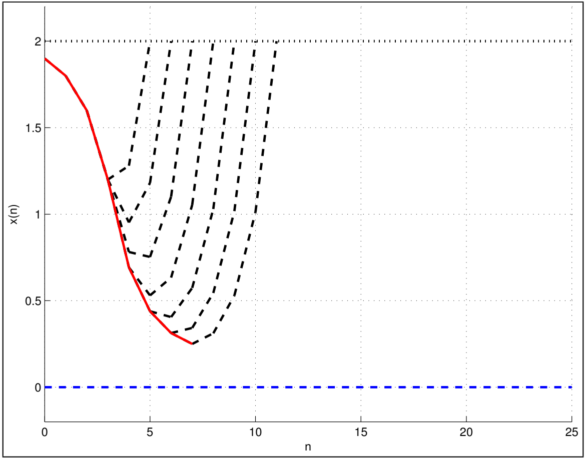

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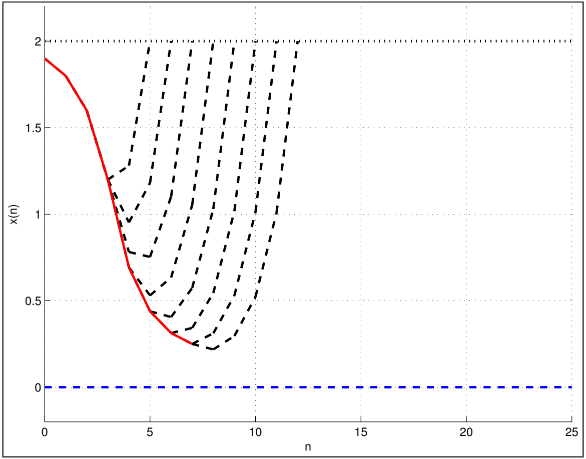

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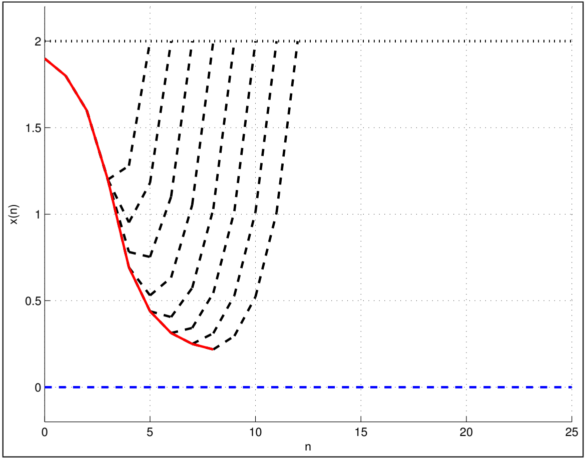

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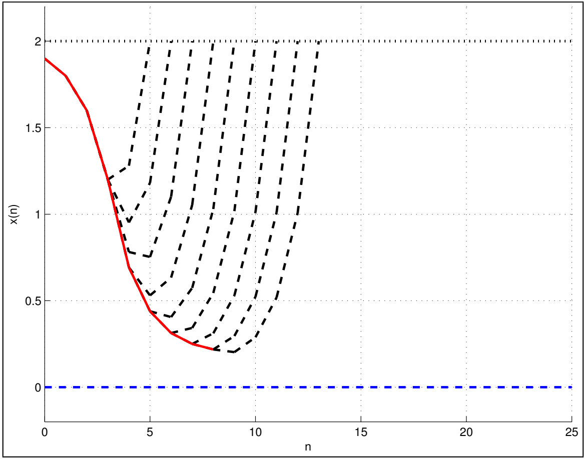

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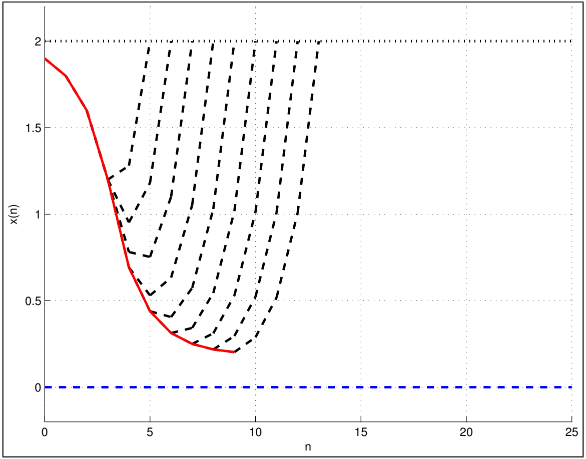

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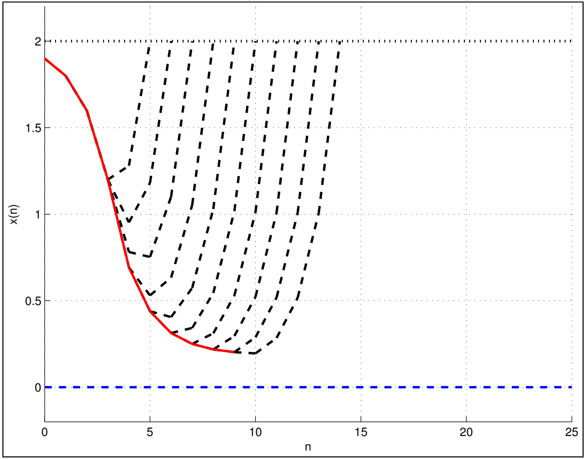

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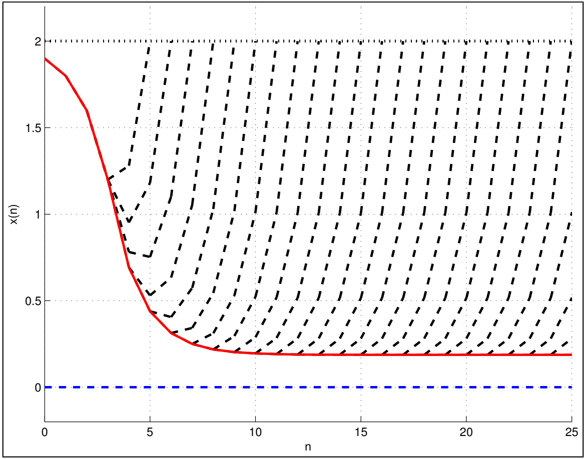

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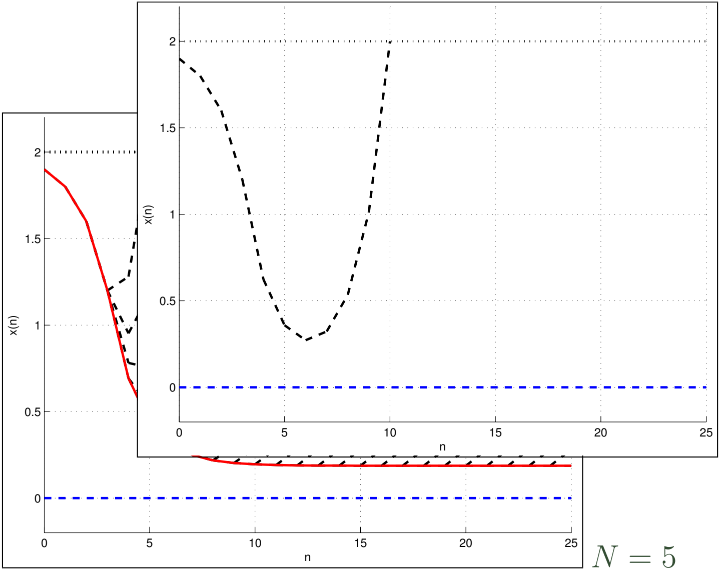

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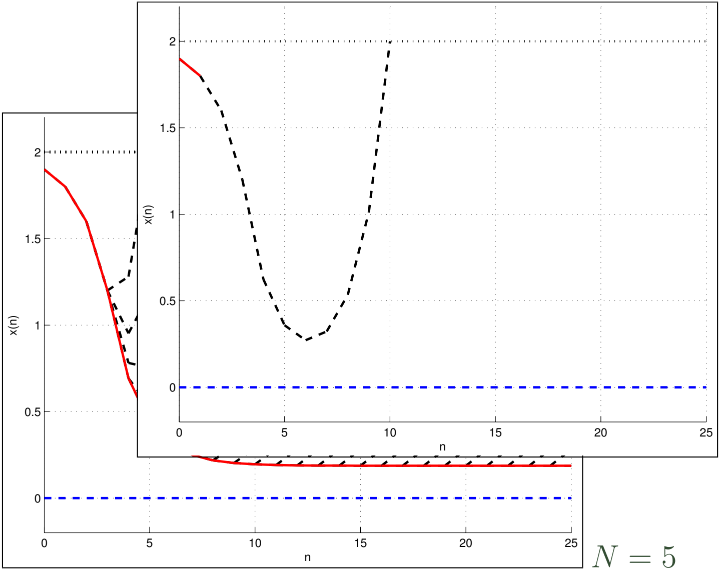

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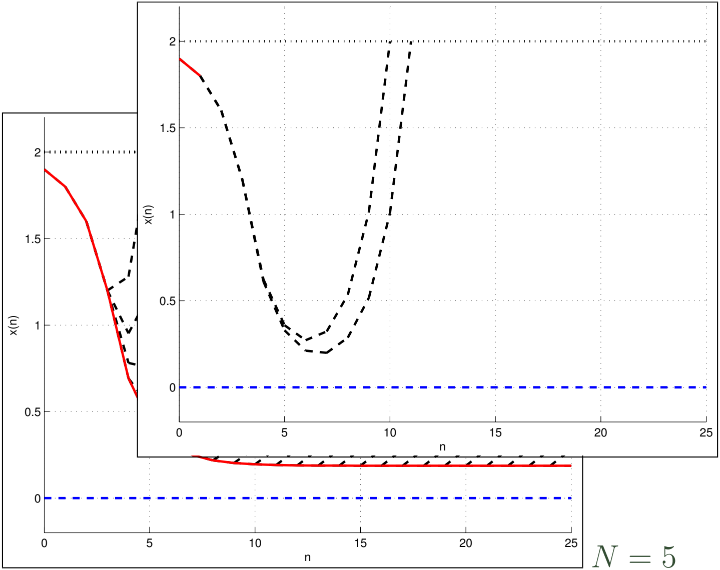

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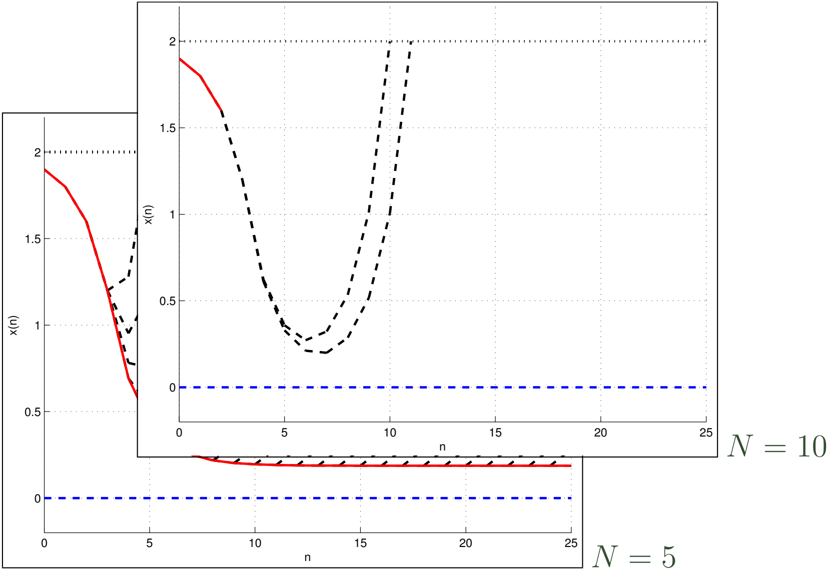

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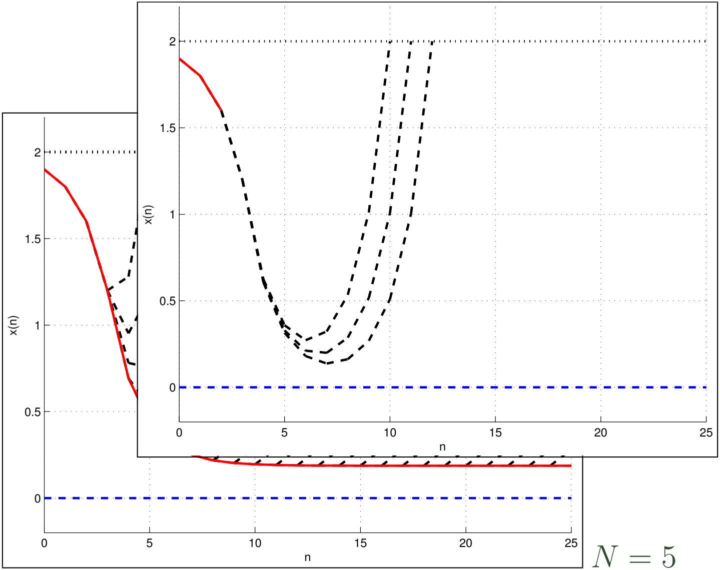
Practical asymptotic stability in Example 1



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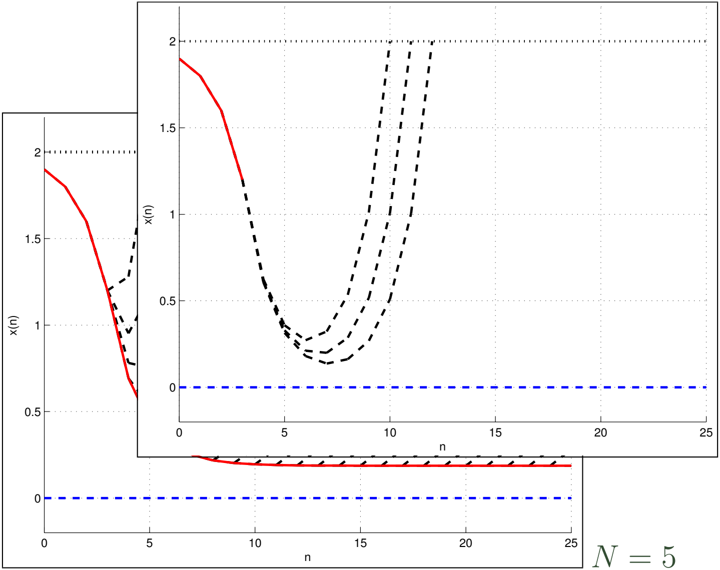

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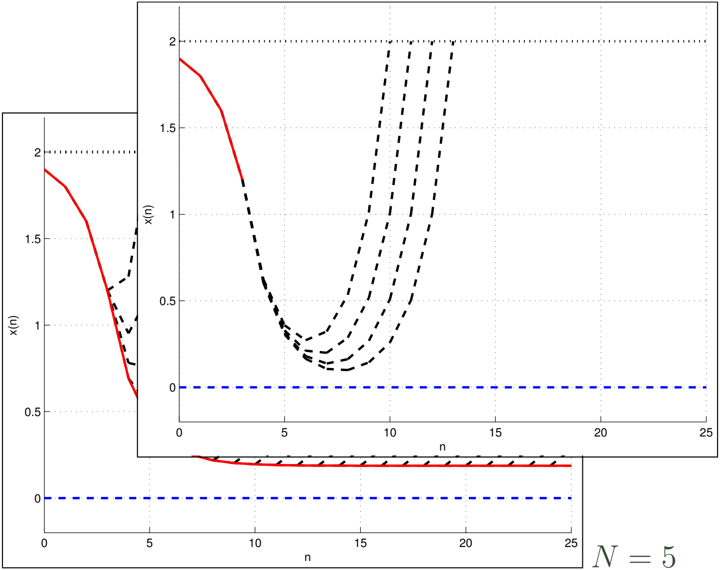

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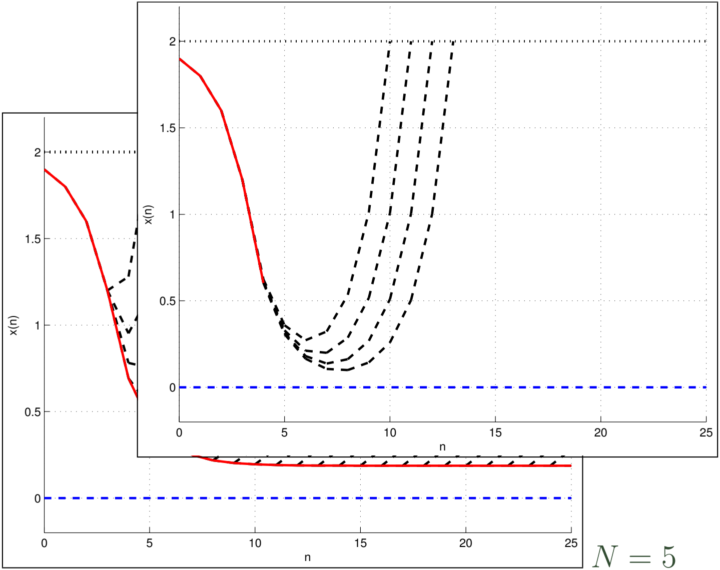

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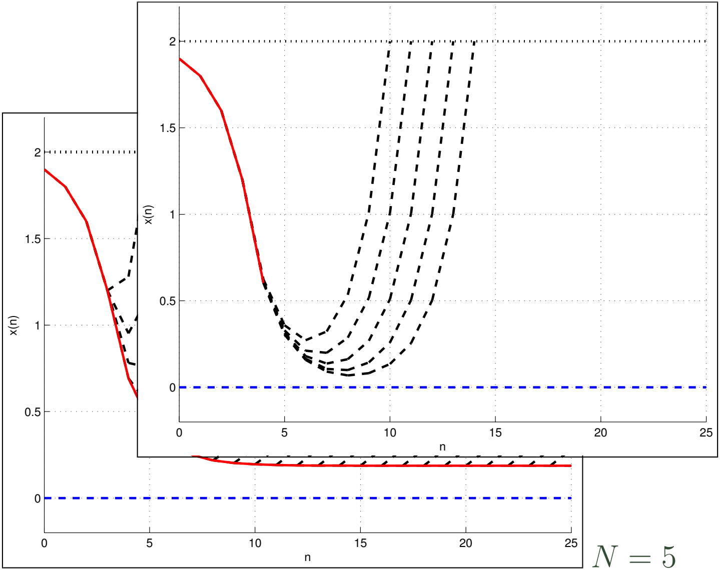

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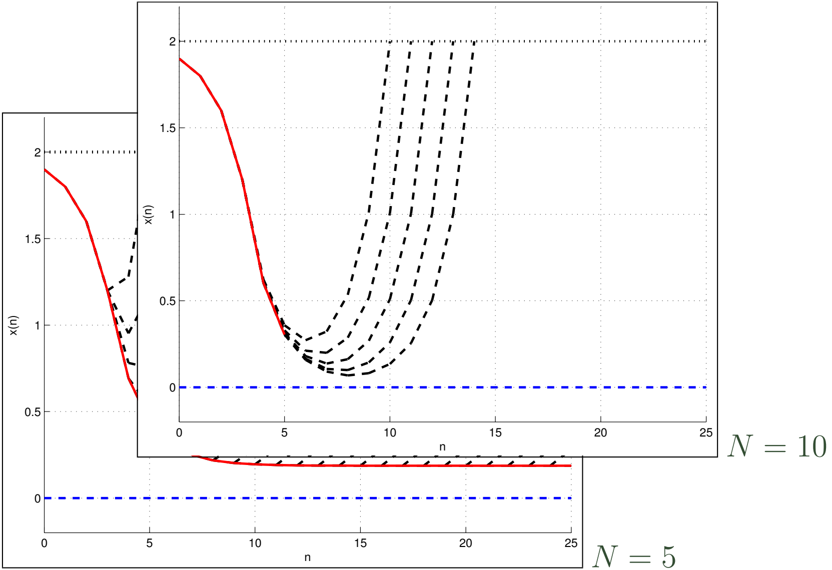

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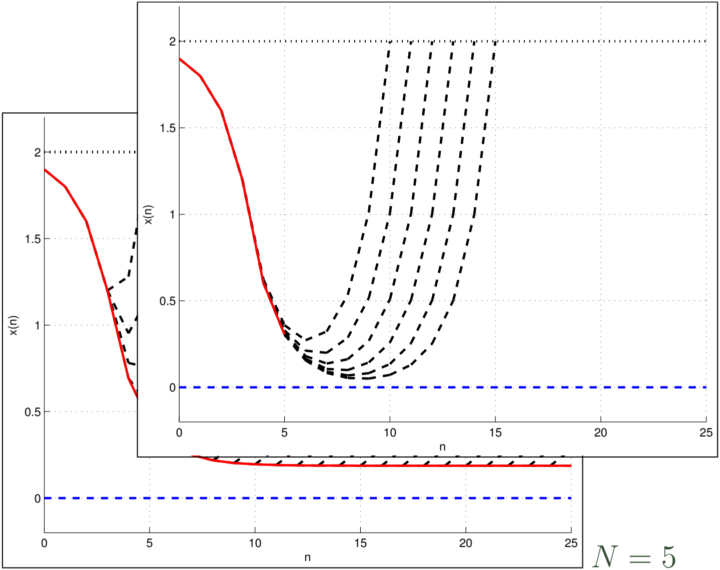

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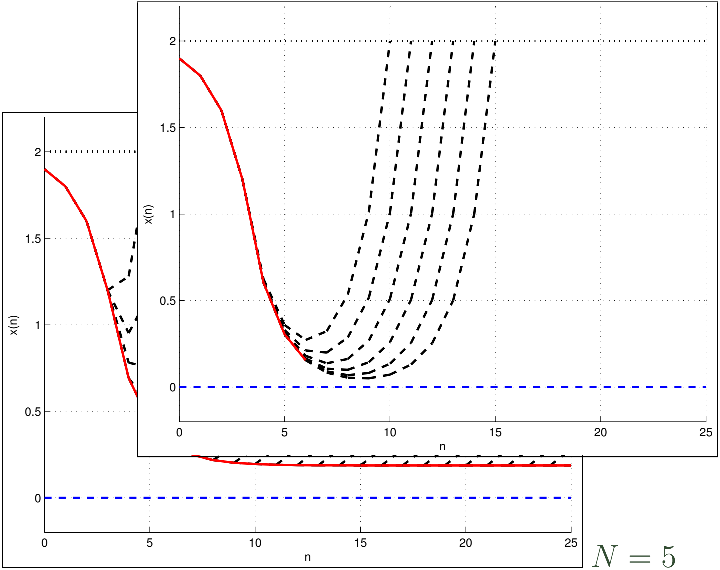

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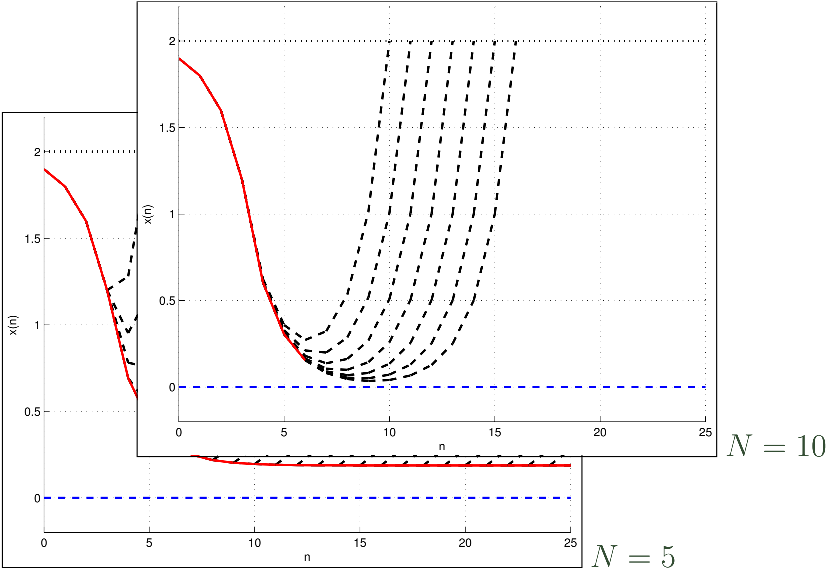

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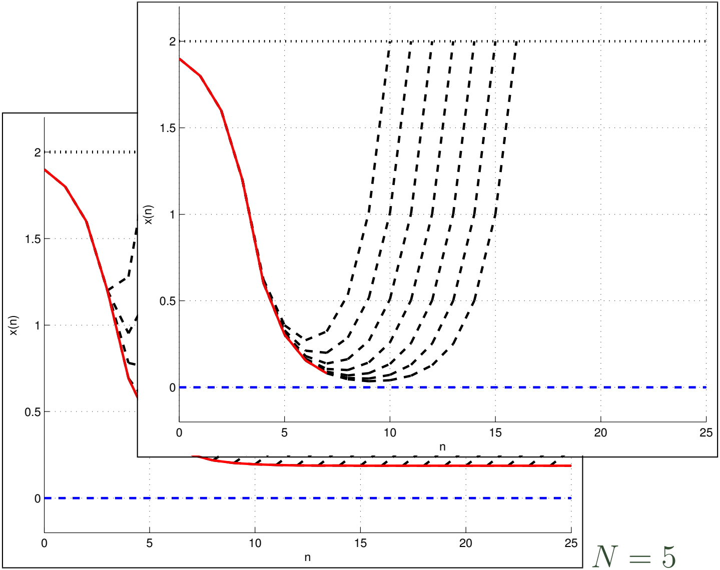

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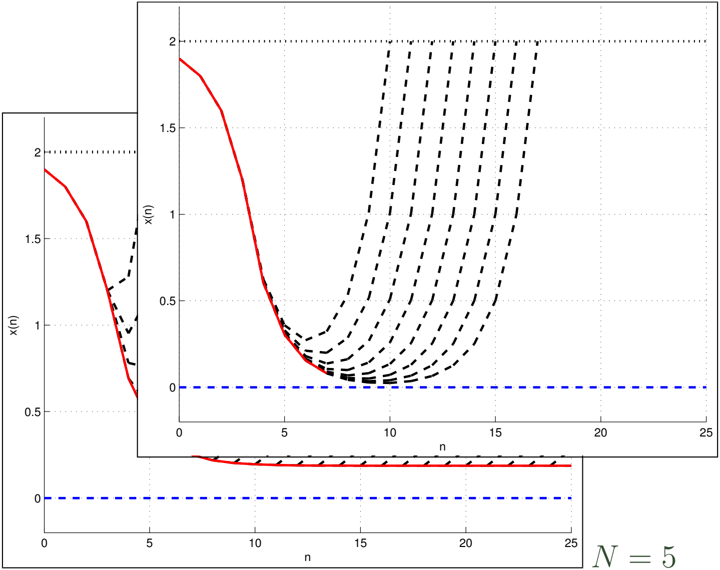

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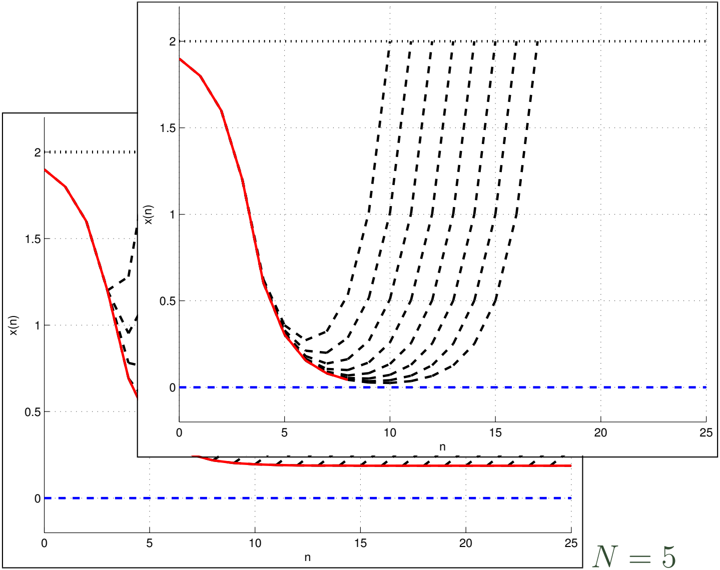

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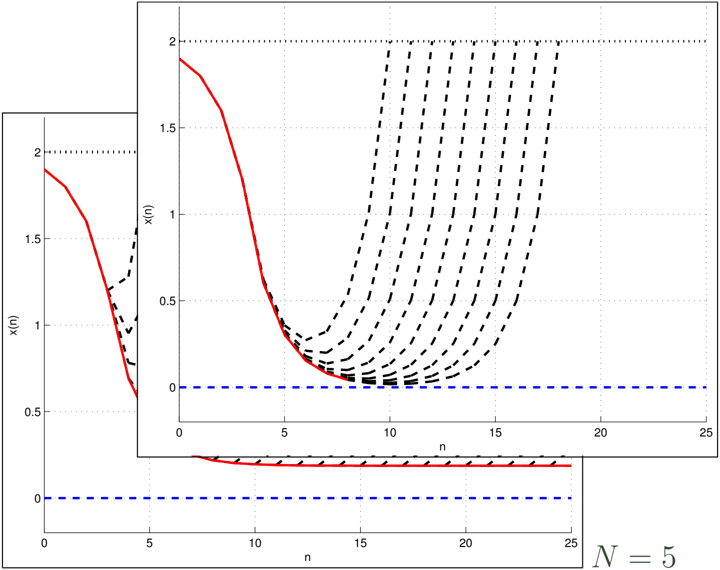

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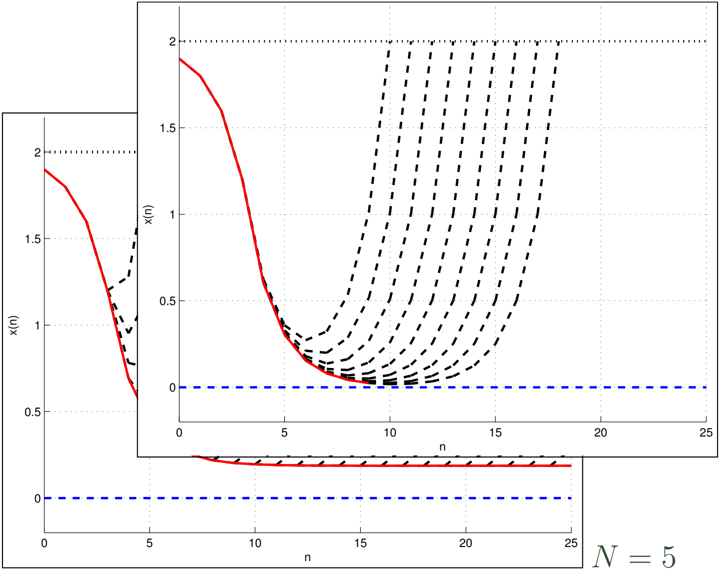

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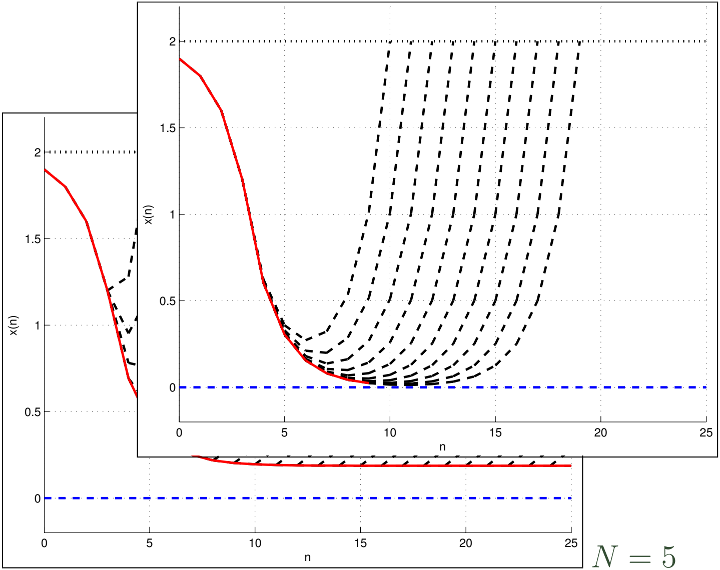

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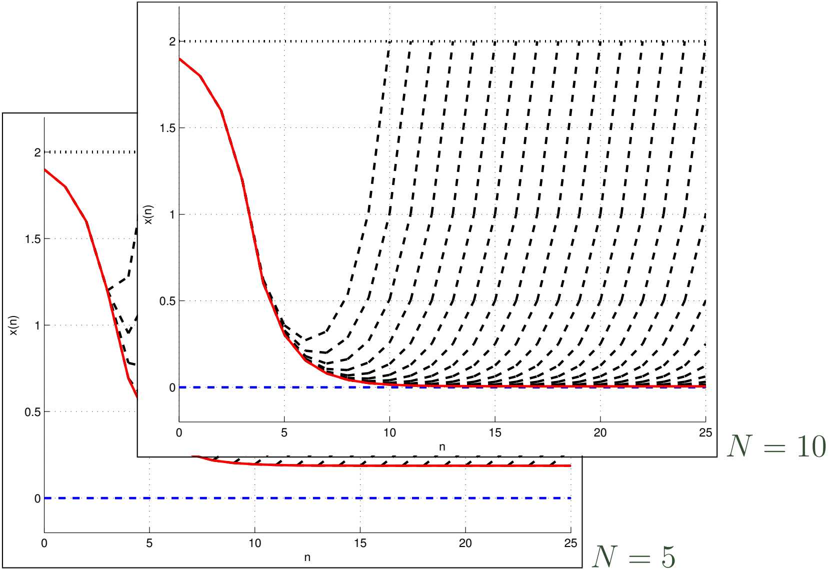

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MPC with terminal constraints

Similar results are available for MPC with terminal conditions
[Angeli/Amrit/Rawlings '12; Diehl/Rawlings '11; Gr./Panin '15]

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Summary

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- Under stabilizability or reachability conditions, it is closely related to strict dissipativity and detectability
- For infinite-dimensional linear evolution equation, it can be seen as a particular instance of a more general sensitivity property
- If the turnpike property holds, then Model Predictive Control provides an approximate solution method for infinite horizon optimal control problems

Literature

Recent Surveys:

Timm Faulwasser and Lars Grüne
Turnpike properties in optimal control: an overview of discrete-time and continuous-time results
https://arxiv.org/abs/2011.13670

Lars Grüne
Dissipativity and optimal control
https://arxiv.org/abs/2101.12606