Two Pathways Lemma and Its Control Applications I

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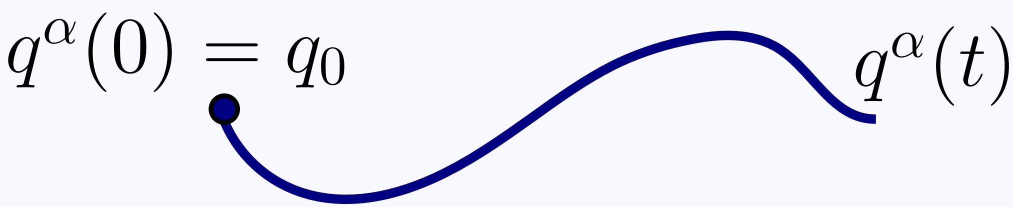
Two Pathways Lemma

Consider dynamical system on manifold ($\alpha \in \mathbb{R}$ - scalar parameter)

$$\frac{d}{dt} q^\alpha(t) = \alpha V_t(q^\alpha(t)),$$



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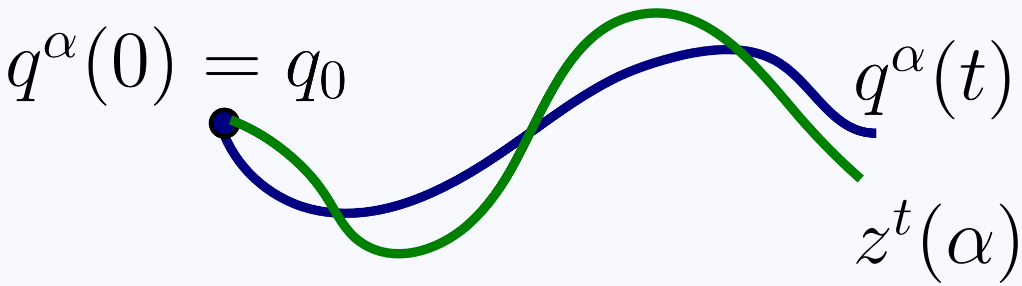

$$\frac{d}{d\alpha} z^t = \sum_{k=1}^{m-1} \alpha^{k-1} \int_0^t dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \dots \int_t^{t_{k-1}} dt_k [V_{t_k}, [\hat{V}_{t_{k-1}}, \dots, [V_{t_2}, V_{t_1}], \dots,],](z^t)$$

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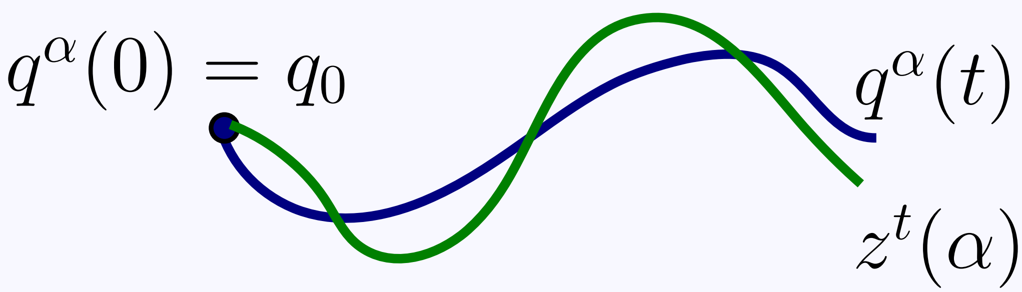

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For vector fields $V(q), W(q)$ with values in tangent space $T_M(q)$ Lie bracket

$$[V, W](q) \in T_M(q)$$

Two Pathways Lemma

If $M = \mathbb{R}^n$

$$V(q) = \begin{bmatrix} V^1(q) \\ \dots \\ V^n(q) \end{bmatrix}, \quad W(q) = \begin{bmatrix} W^1(q) \\ \dots \\ W^n(q) \end{bmatrix}$$

then

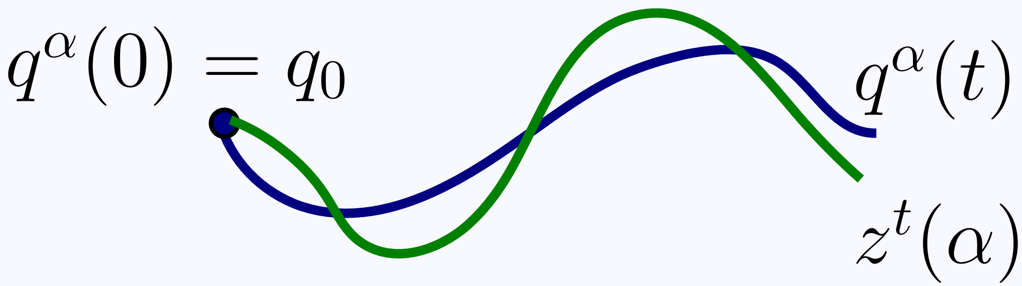
$$[V, W](q) := \begin{bmatrix} \sum_{i=1}^n V^i(q) \frac{\partial W^1}{\partial q^i}(q) - W^i(q) \frac{\partial V^1}{\partial q^i}(q) \\ \dots \\ \sum_{i=1}^n V^i(q) \frac{\partial W^n}{\partial q^i}(q) - W^i(V) \frac{\partial V^n}{\partial q^i}(q) \end{bmatrix}$$

IMPORTANT: Lie bracket of vector-fields is again a vector-field

Two Pathways Lemma

Consider dynamical system on manifold ($\alpha \in \mathbb{R}$ - scalar parameter)

$$\frac{d}{dt} q^\alpha(t) = \alpha V_t(q^\alpha(t)),$$



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solution $q^\alpha(t)$ is approximated by solution $z^t(\alpha)$

$$\|q^\alpha(t) - z^t(\alpha)\| \quad \text{is small}$$

Two Pathways Lemma

TWO PATHWAYS LEMMA:
For any $q_0 \in M$, function $\psi \in C^m(M,E)$ there exists a constant K, θ_0 and neighbourhood $\mathcal{O}$ of q_0 such that for all sufficiently small α and $t \in [0,\theta_0)$ and any solutions $q^\alpha(t)$ and $z^t(\alpha)$ with initial conditions $z^t(0) = q(0) \in \mathcal{O}$

$$\|\psi(q^\alpha(t)) - \psi(z^t(\alpha))\| < K\alpha^m$$

$$\frac{d}{dt} q^\alpha(t) = \alpha V_t(q^\alpha(t))$$

$$\frac{d}{d\alpha} z^t = \sum_{k=1}^{m-1} \alpha^{k-1} \int_0^t dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \ldots \int_t^{t_{k-1}} dt_k [V_{t_k}, [\widehat{V}_{t_{k-1}}, \ldots, [V_{t_2}, V_{t_1}], \ldots,],](z^t)$$

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- Prehistory: Derivative of matrix exponential and Wilcox formula

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- Mathematical tools: Extension of Chronological Calculus by R.Kipka and Yu.L.
- Two Pathways Lemma and Applications

Derivative of Matrix exponential and Wilcox Formula

Let $A : \mathbb{X} \to \mathbb{X}$ be linear bounded operator (or matrix $A \in \mathbb{R}^{n\times n}$)
Operator (matrix) exponential

$$e^A := I + A + \frac{A^2}{2!} + \ldots + \frac{A^m}{m!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Find derivative of the mapping $A \to e^A$

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Find derivative of the mapping $A \to e^A$

F.Schur, H.Poincare

$$de^A = e^A \, \frac{I - e^{-\mathrm{ad}(A)}}{\mathrm{ad}(A)}$$

Commutator $[A, B] := AB - BA$, linear operator $\mathrm{ad}(A)$

$$\mathrm{ad}(A)B := [A, B], \quad (\mathrm{ad}(A))^k B := [A, (\mathrm{ad}(A))^{k-1} B],$$

$k = 1, 2, \ldots$.

Derivative of Matrix exponential and Wilcox Formula

F.Schur, H.Poincare

$$de^A = e^A \, \frac{I - e^{-\mathrm{ad}(A)}}{\mathrm{ad}(A)}$$

Substitute x by $\mathrm{ad}(A)$ in

$$\frac{1-e^{-x}}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} x^k$$

Derivative of Matrix exponential and Wilcox Formula

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How to derive formula $(\square)$

Derivative of Matrix exponential and Wilcox Formula

In modern textbooks
see **Rossmann,** ***Lie Groups, 2002***
they use formula from
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Wilcox (1967)

$$\frac{\partial}{\partial \alpha} e^{tH(\alpha)} = \int_0^t e^{(t-s)H(\alpha)} H'(\alpha) e^{sH(\alpha)} \, ds$$

Derivative of Matrix exponential and Wilcox Formula

How to prove

Wilcox (1967)

$$\frac{\partial}{\partial \alpha} e^{tH(\alpha)} = \int_0^t e^{(t-s)H(\alpha)} H'(\alpha) e^{sH(\alpha)} \, ds$$

Use formula for differentiation of solution of **LINEAR** differential equation by parameter

Let $Y_{\Delta\alpha}(t) := e^{tH(\alpha+\Delta\alpha)}$ be a fundamental solution

$$Y'_{\Delta\alpha}(t) = H(\alpha + \Delta\alpha) Y_{\Delta\alpha}(t), \quad Y_{\Delta\alpha}(0) = I$$

and $Y_0(t) := e^{tH(\alpha)}$ be a solution

$$Y'_0(t) = H(\alpha) Y_0(t), \quad Y_0(0) = I$$

Derivative of Matrix exponential and Wilcox Formula

Then

$$Y'_{\Delta\alpha}(t) = H(\alpha)Y_{\Delta\alpha}(t) + \Delta H(\Delta\alpha)Y_{\Delta\alpha}(t)$$

where

$$\Delta H(\Delta\alpha) := H(\alpha + \Delta\alpha) - H(\alpha)$$

Use Cauchy formula

$$Y_{\Delta\alpha}(t) = Y_0(t) + \int_0^t e^{(t-s)H(\alpha)}\Delta H(\Delta\alpha)Y_{\Delta\alpha}(s)\, ds$$

This implies
$$\frac{\partial}{\partial\alpha}e^{tH(\alpha)} = \int_0^t e^{(t-s)H(\alpha)}H'(\alpha)e^{sH(\alpha)}\, ds$$

Generalization of Wilcox formula for vector fields on manifolds

Derivative of Matrix exponential and Wilcox Formula

We have

$$\int_0^t e^{(t-s)H(\alpha)} H'(\alpha) e^{sH(\alpha)}\, ds = e^{tH(\alpha)} \int_0^t e^{-sH(\alpha)} H'(\alpha) e^{sH(\alpha)}\, ds$$

But

$$\left(e^{tA} B e^{-tA}\right)^{(k)} = e^{tA} (\mathrm{ad}(A))^k B e^{-tA}$$

This implies

$$\int_0^t e^{-sH(\alpha)} H'(\alpha) e^{sH(\alpha)}\, ds = \int_0^t e^{-s\ \mathrm{ad}(H(\alpha))} H'(\alpha)\, ds$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

M is *Banach manifold* of class C^m over a Banach space E is a paracompact Hausdorff space M along with a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha) \,:\, \alpha \in A\}$, where A is an indexing set. Consider dynamical system on M

$$\dot{q} = V_t(q), \quad q(t_0) = q_0.$$

With each nonautonomous vector field V_t on M, we associate a local flow

$$P_{t_0,t}(q_0) := q(t; t_0, q_0)$$

Consider the vector space $C^m(M, E)$ of C^m functions $\varphi : M \to E$ Then diffeomorphism $P : M \to M$ define a linear operator

$$\widehat{P}(\varphi) := \varphi \circ P \quad \Rightarrow \quad \widehat{P}(\varphi)(q) = \varphi(P(q))$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

Vector field V on M defines linear operator
$\widehat{V}: C^m(M,E) \to C^{m-1}(M,E)$

$$\widehat{V}(\varphi)(q) := \varphi_*(q)V(q)$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

Vector field V on M defines linear operator
$\widehat{V} : C^m(M,E) \to C^{m-1}(M,E)$

$$\widehat{V}(\varphi)(q) := \varphi_*(q)V(q)$$

Consider operator-valued function $t \to A_t$ whose values are linear mappings $A_t : C^m(M,E) \to C^p(M,E)$
Then define integral of such function

$$\left(\int_{t_0}^{t_1} A_\tau \, d\tau\right)(\varphi)(q) := \int_{t_0}^{t_1} A_\tau(\varphi)(q)\, d\tau$$

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$$\left(\int_{t_0}^{t_1} A_\tau \, d\tau\right)(\varphi)(q) := \int_{t_0}^{t_1} A_\tau(\varphi)(q)\, d\tau$$

Then we have integral equation

$$\widehat{P}_{t_0,t} = \widehat{Id} + \int_{t_0}^{t} \widehat{P}_{t_0,\tau} \circ \widehat{V}_\tau \, d\tau \quad \Rightarrow \quad \frac{d}{dt}\widehat{P}_{t_0,t} = \widehat{P}_{t_0,t} \circ \widehat{V}_t, \qquad \widehat{P}_{t_0,t_0} = \widehat{Id}.$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

Integral differential equations

$$\widehat{P}_{t_0,t} = \widehat{Id} + \int_{t_0}^{t} \widehat{P}_{t_0,\tau} \circ \widehat{V}_\tau \, d\tau \quad \Rightarrow \quad \frac{d}{dt}\widehat{P}_{t_0,t} = \widehat{P}_{t_0,t} \circ \widehat{V}_t, \qquad \widehat{P}_{t_0,t_0} = \widehat{Id}.$$

PRODUCT RULE (**Kipka & Yu.L.(2015)**)

Let

$$\widehat{P}_t = \widehat{P}_{t_0} + \int_{t_0}^{t} \frac{d}{d\tau}\widehat{P}_\tau \, d\tau, \qquad \widehat{Q}_t = \widehat{Q}_{t_0} + \int_{t_0}^{t} \frac{d}{d\tau}\widehat{Q}_\tau \, d\tau,$$

then

$$\int_{t_1}^{t_2} \frac{d}{dt}(\widehat{P}_t \circ \widehat{Q}_t) \, dt = \int_{t_1}^{t_2} (\frac{d}{dt}\widehat{P}_t \circ \widehat{Q}_t + \widehat{P}_t \circ \frac{d}{dt}\widehat{Q}_t) \, dt.$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

Original Chronological Calculus is due to **Agrachev & Gamkrelidze (mid-1970s)**

Let V be a vector field and $F : M \to M$ be a C^m diffeomorphism.

Define the operator $\mathrm{Ad}(\widehat{F}) : \widehat{V} \mapsto \widehat{F} \circ \widehat{V} \circ \widehat{F^{-1}}$.

Recall Lie bracket $[V, W]$ of vector fields V and W is the vector field whose operator representation has form

$$\widehat{[V, W]} = \widehat{V} \circ \widehat{W} - \widehat{W} \circ \widehat{V}$$

Define an operator $\mathrm{ad}\,\widehat{V}_t$ by

$$(\mathrm{ad}\,\widehat{V}_t) \circ \widehat{W}_t = \left[\widehat{V}_t, \widehat{W}\right].$$

Extension of Chronological Calculus by R.Kipka & Yu.L.

$$\mathrm{Ad}(\widehat{F}) : \widehat{V} \mapsto \widehat{F} \circ \widehat{V} \circ \widehat{F^{-1}}$$

$$(\mathrm{ad}\, \widehat{V}_t) \circ \widehat{W} = \left[\widehat{V}_t, \widehat{W}\right].$$

Then important result

$$\frac{d}{dt}\mathrm{Ad}(\widehat{P}_{t_0,t}) \circ \widehat{W} = \mathrm{Ad}(\widehat{P}_{t_0,t}) \circ \mathrm{ad}\, \widehat{V}_t \circ \widehat{W}$$

Generalized Wilcox Formula for Manifolds

Consider a parametric family of dynamical systems on Banach manifold M

$$q'(t) = V(t, q(t), \alpha)$$

where α is a scalar parameter
Let

$$V(t, q, \alpha + \Delta\alpha) = V(t, q, \alpha) + \Delta\alpha W(t, q) + U(t, q, \alpha, \Delta\alpha)$$

where

$$\lim_{\Delta\alpha \to 0} \frac{1}{\Delta\alpha} \int_{t_1}^{t_2} \|\psi_*(q')U(s, q', \alpha, \Delta\alpha)\| \, ds$$

Generalized Wilcox Formula for Manifolds

Generalized Wilcox Formula

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \int_0^t \mathrm{Ad}(\widehat{P}_s^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha \, ds \circ \widehat{P}_t^\alpha$$

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \int_0^t \mathrm{Ad}(\widehat{P}_{t,s}^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha \, ds$$

Generalized Wilcox Formula for Manifolds

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$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \int_0^t \mathrm{Ad}(\widehat{P}_s^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha \, ds \circ \widehat{P}_t^\alpha$$

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \int_0^t \mathrm{Ad}(\widehat{P}_{t,s}^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha \, ds$$

PROOF. USE Product Rule

$$\widehat{P}_t^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_t^\alpha\right)^{-1} - \mathrm{Id} = \int_0^t \frac{d}{ds}\widehat{P}_s^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_s^\alpha\right)^{-1} \, ds =$$

$$= \int_0^t (\widehat{P}_s^{\alpha+\Delta\alpha} \circ \widehat{V}_s^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_s^\alpha\right)^{-1} - \widehat{P}_s^{\alpha+\Delta\alpha} \circ \widehat{V}_s^\alpha \left(\widehat{P}_s^\alpha\right)^{-1}) \, ds$$

Generalized Wilcox Formula for Manifolds

Then we obtain

$$\widehat{P}_t^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_t^{\alpha}\right)^{-1} - \mathrm{Id} = \Delta\alpha \int_0^t \widehat{P}_s^{\alpha} \circ \widehat{W}_s \left(\widehat{P}_s\right)^{-1} \, ds \circ \widehat{P}_t^{\alpha} + \int_0^t \widehat{R}_s \, ds$$

and

$$\widehat{P}_t^{\alpha+\Delta\alpha} = \widehat{P}_t^{\alpha} + \Delta\alpha \int_0^t \mathrm{Ad}(\widehat{P}_s^{\alpha}) \circ \widehat{W}_s \, ds \circ \widehat{P}_s^{\alpha} + \hat{o}_t(\Delta\alpha)$$

This implies

$$\frac{\partial}{\partial\alpha} \widehat{P}_t^{\alpha} = \int_0^t \mathrm{Ad}(\widehat{P}_s^{\alpha}) \circ \frac{\partial}{\partial\alpha} \widehat{V}_s^{\alpha} \, ds \circ \widehat{P}_t^{\alpha}$$

Generalized Wilcox Formula for Manifolds

Functional-differential equation

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \int_0^t \mathrm{Ad}(\widehat{P}_{t,s}^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha \, ds$$

Two Pathways Lemma: Approximation Result

Consider differential equation on M

$$\frac{d}{dt}\, q = \alpha V_t(q)$$

Two Pathways Lemma: Approximation Result

Consider differential equation on M

$$\frac{d}{dt}\, q = \alpha V_t(q)$$

Corresponding operator differential equation

$$\frac{d}{dt}\widehat{P}_t^{\alpha} = \widehat{P}_t^{\alpha} \circ \alpha \widehat{V}_t$$

Two Pathways Lemma: Approximation Result

Consider differential equation on M

$$\frac{d}{dt}\,q = \alpha V_t(q)$$

Corresponding operator differential equation

$$\frac{d}{dt}\widehat{P}^{\alpha}_t = \widehat{P}^{\alpha}_t \circ \alpha\widehat{V}_t$$

From generalized Wilcox formula

$$\frac{d}{d\alpha}\widehat{P}^{\alpha}_t = \widehat{P}^{\alpha}_t \circ \int_0^t \mathrm{Ad}(\widehat{P}^{\alpha}_{t,s}) \circ \widehat{V}_s\, ds$$

This is NOT operator differential equation.
This is operator functional-differential equation

Two Pathways Lemma: Approximation Result

Solution

$$\frac{d}{d\alpha}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \int_0^t \mathrm{Ad}(\widehat{P}_{t,s}^\alpha) \circ \widehat{V}_s \, ds$$

can be approximated by solution of operator differential equation

$$\frac{d}{d\alpha}\widehat{Z}_t^\alpha = \widehat{Z}_t^\alpha \circ \widehat{F}_t^\alpha, \qquad \widehat{Z}_t^\alpha\Big|_{\alpha=0} = \widehat{Id}$$

where

$$\widehat{F}_t^\alpha := \sum_{k=1}^{m-1} \alpha^{k-1} \int_0^t dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \ldots \int_t^{t_{k-1}} dt_k [\widehat{V}_{t_k}, [\widehat{V}_{t_{k-1}}, \ldots, [\widehat{V}_{t_2}, \widehat{V}_{t}$$

Approximation means

$$\widehat{P}_t^\alpha = \widehat{Z}_t^\alpha + \widehat{O}(\alpha^m),$$

Two Pathways Lemma: Approximation Result

In the proof to evaluate the vector field

$$\int_0^t \mathrm{Ad}(\widehat{P}^{\alpha}_{t,t_1}) \circ \widehat{V}_{t_1}\ dt_1$$

we use relation

$$\mathrm{Ad}(\widehat{P}_{t,t_1}) \circ \widehat{W} = \widehat{W} + \alpha \int_t^{t_1} dt_2\ \mathrm{Ad}(\widehat{P}_{t,t_2}) \circ \mathrm{ad}(\widehat{V}_{t_2}) \circ \widehat{W}$$

SMALL TIME

Consider

$$\frac{d}{dt}\, q(t) = V_t(q(t)), \quad q(0) = q_0$$

Applications to Asymptotic Behaviour

SMALL TIME

Consider

$$\frac{d}{dt}\, q(t) = V_t(q(t)), \quad q(0) = q_0$$

Then solution

$$\frac{d}{d\alpha} z(\alpha) = G_T^\alpha(z(\alpha)), \quad z(0) = q_0$$

$$G_T^\alpha(z) := \sum_{k=1}^{m-1} \alpha^{k-1} \int_0^1 d\tau_1 \int_1^{\tau_1} d\tau_2 \ldots \int_1^{\tau_{k-1}} d\tau_k [V_{T\tau_k}, [V_{T\tau_{k-1}}, \, ldots, [V_{T\tau_2}$$

approximates $q(T)$ for small T

$$\|\psi(q(T)) - \psi(z(T))\| < KT^m$$

Applications to Asymptotic Behaviour

LARGE TIME

Consider

$$\frac{d}{dt}\,q(t) = V_t(q(t)), \quad q(0) = q_0$$

Then solution

$$\frac{d}{d\alpha}z(\alpha) = G_T^{\alpha}(z(\alpha)), \quad z(0) = q_0$$

$$\sum_{k=1}^{m-1} \alpha^{k-1} \int_0^{T^2} d\tau_1 \int_{T^2}^{\tau_1} d\tau_2 \ldots \int_{T^2}^{\tau_{k-1}} d\tau_k [V_{\tau_k/T}, [V_{\tau_{k-1}/T}, \ldots, [V_{\tau_2/T}, V_{\tau_1/T}], \ldots$$

Approximates $q(T)$ for large T

$$\|\psi(q(T)) - \psi(z(\alpha_*))\| < K(\frac{1}{T})^m$$