Optimization Methods in Bank Balance Sheet Managemant

an Industry Academic Collaboration
Midwest Optimization Seminar, Grand Valley State University

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Dedicated to

Asen Dontchev

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Bank Balance Sheet Managemant Problem

A bank balance sheet consists of many assets and liabilities.
Task: Maintaining a diverse portfolio to increase shareholder benefits while limiting risks.
Modeling: Constrained Optimization Problems.

Bank Balance Sheet Managemant

- This is an industry-academic collaboration.
- We found that many methods in multi-objective and convex optimization are highly relevant and provide insight for practitioners.
- On the other hand, practical considerations often lead to interesting new directions of investigation.

Related Theories in Financial Math

- Portfolio Theory
- Hedging and Derivative Pricing
- Insurance and Extreme Events

Markowitz Portfolio Theory

Idea: Trade-off between expected return and risk measured by variance.
Users: Mutual funds, Hedge funds and Banks
Math: Linear-Quadratic Optimization Model

Ref: H. Markowitz, Portfolio Selection, Cowles Monograph, Vol. 16, Wiley, N.Y. 1959.

Growth Optimal Portfolio Theory

Idea: Maximizing a log utility function. Supposedly taking care of both reward and risk.
Users: Mutual funds, Hedge funds and Banks
Math: Convex optimization

Ref: L. C. McLean, E.O. Thorpe, W.T. Ziemba, The Kelly Capital Growth Investment criterion, theory and practice, World Scientific, Singapore, 2011

Hedging and Derivative Pricing

Idea: Using dynamical hedging one can replicate the behavior of a financial derivative with the underlying and, therefore, derive its ‘fair’ price.
Users: Investment Banks, Banks and Insurance companies
Math: Stochastic Differential Equations and Feymann-Kac formula

Ref: F. Black and M. Scholes, The price of options and corporate liabilities, J. Polit. Econ. 81, 637-645, (1973)

Insurance and Extreme Events

Idea: Pay attention to the consequence of rare but extreme events.
Users: Insurance company, banks and government regulatory and planning agencies.
Math: Extreme value theory.

Ref: A. J. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management, Princeton University Press, 2015

Growth Optimal Portfolio (GOP) Theory

The first line of our investigation follows the idea of GOP. Find a portfolio that maximize the log utility so as to maximize the growth. Problem: too risky.

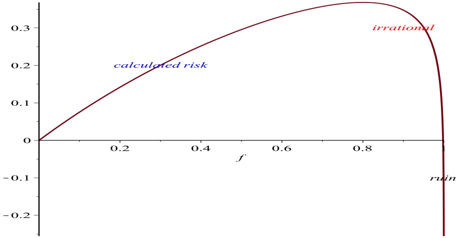


Figure: Log return curve

Modified Growth Optimal Portfolio Theory

The main reason why GOP is too risky is that it assumes investing in infinite horizon. Considering total return in a finite investment horizon, we have a bell shaped curve:

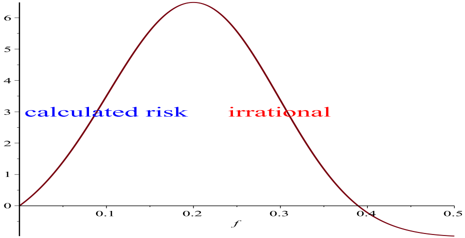


Figure: Return curve

Modified Growth Optimal Portfolio Theory

Two alternative optimal points appears: the best return/leverage ratio and the inflection point.

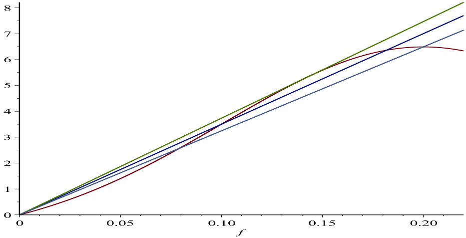


Figure: Finite investment horizon

Ref: R. Vince and Q. J. Zhu, Optimal betting sizes for the game of Blackjack, Risk J. Portf. Mnag. 4, 53-75 (2015)
M. Lopez de Prado, R. Vince and Q. J. Zhu, Optimal risk budgeting under a finite investment horizon, MDPI Risks, 7(3), 86, (2019).

Modified Growth Optimal Portfolio Theory

Applying to bank balance sheet problem these alternative optimal portfolios provide better balance between return and risk.

On going further investigation is focusing on replacing return/leverage ratio by return/risk ratio and sensitivity.

Ref: S. Dewasurendra, P. Júdice and Q. J. Zhu, The optimal leverage level of the banking sector, Risks, MDPI 7(2), 51 (2019)

A General Framework for Portfolio Theory

Another line of investigation follows the idea of Markowitz portfolio theory: trade-off between utility $\mathfrak{u} : A \mapsto \mathbb{R}$ and vector valued risk function $\mathfrak{r} : A \mapsto \mathbb{R}^M$ defined on the set of addmissible portfolios A.

Efficient Portfolio

A portfolio $x^* \in A$ is Pareto efficient provided that there does *not exist* any $x' \in A$ such that either

$$[\mathfrak{r}(x') \leq \mathfrak{r}(x^*) \quad \text{and} \quad \mathfrak{u}(x') > \mathfrak{u}(x^*)]$$

or

$$[\mathfrak{r}(x') \leq \mathfrak{r}(x^*), \mathfrak{r}(x') \neq \mathfrak{r}(x^*), \quad \text{and} \quad \mathfrak{u}(x') \geq \mathfrak{u}(x^*)]$$

holds.

A General Framework for Portfolio Theory

We focused on the Pareto efficient frontier in the risk-reward space and relate it to the corresponding efficient portfolios.

Efficient Frontier

$\mathcal{G}_{\text{eff}} = \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) := \{(r, \mu) \,:\,$ there exists an efficient portfolio $x^* \in A$ with $\mathfrak{r}(x^*) = r$ and $\mathfrak{u}(x^*) = \mu\}$.

Efficient frontier: one risk

Key results: The Pareto efficient set is a connected set that can be represented using the graphs of a combination of convex and concave functions in the risk-reward space, which maps continuously to efficient portfolios.

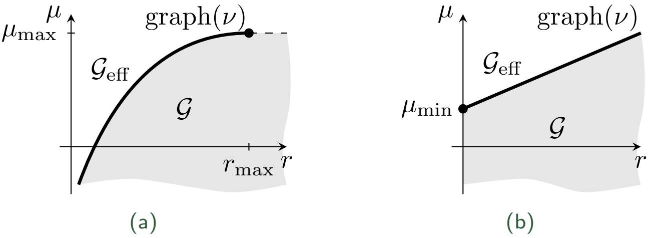


Figure: $\mathcal{G}_{\text{eff}}$ examples with one risk

Ref: S. Maier-Paape, P. Júdice, A. Platen and Q. J. Zhu, Scalar and Vector Risk in the General Framework of Portfolio Theory, working book..

Efficient frontier: one risk

Graph (b) corresponds to

Two fund separation theorem

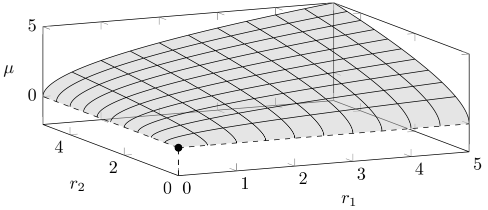
Trade-off between expected return and variance, the efficient frontier is a straight line that passes through two points: a risk free bond and a "market portfolio" of risky assets. Thus, all efficient portfolios can be derived as a linear combination of the two.

This is the theoretical foundation for index investing. It shows why properties of the efficient frontier matter.

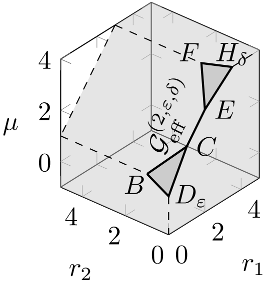
Ref: J. Tobin, Liquidity preference as behavior towards risk. Review of Economic Studies, 25(1):65-86, (1958)

Efficient frontier: multiple-risks

Complications arise when there are multiple risks: moving between efficient portfolios becomes harder.



(a) $\mathcal{G}_{\text{eff}}$ not closed.



(b) $\mathcal{G}_{\text{eff}}$ projection not convex.

Figure: $\mathcal{G}_{\text{eff}}$ with multiple risks.

Linear Model

Applying the general framework to the concrete problem of bank balance sheet requires to specify the reward (utility) and risk functions. The simplest case is to assume that they are all linear. We illustrate this with a case study involving interest and credit risks with related information summarized in the table below:

Asset	allocation	returns	interest risk	credit risk
General	x_k	r_k	i_k	c_k
Bonds	y_j	R_j	I_j	none
Fed fund	y_0	R_0	none	none

Table: Notation

Ref. P. Júdice and Q. J. Zhu, Bank Balance Sheet Allocation, .Journal of Banking and Finance, Vol. 133, 2021

Linear Progrmming Model: Primal

Based on the information in the table, we can form a linear programming problem

$$
\begin{array}{rc}
p = \max & \sum_{k=1}^{n}(r_k - R_0)x_k + \sum_{j=-s}^{l} sgn(j)(R_j - R_0)y_j + R_0 e \\
\text{subject to} & \\
\lambda^i & \sum_{k=1}^{n} i_k x_k + \sum_{j=-s}^{l} sgn(j) I_j y_j \leq i \\
\lambda^c & \sum_{k=1}^{n} c_k x_k \leq c,
\end{array}
$$

where long bonds are represented with $j > 0$ and short bonds with $j < 0$. The λ's are the dual variables.

Linear Programming Model: Dual

$$\begin{aligned} d = \min \quad & i\lambda^i + c\lambda^c \\ \text{subject to} \quad & \\ x_k \quad & i_k\lambda^i + c_k\lambda^c \geq r_k - R_0, k = 1, \ldots, n \\ y_j \quad & sgn(j)I_j\lambda^i \geq sgn(j)(R_j - R_0), j = -s, \ldots, l \\ & \lambda^i, \lambda^c \geq 0. \end{aligned}$$

The constriants corresponding to y_j (bonds) gives us

$$u := \min_{j=1,\ldots,s} \frac{R_{-j} - R_0}{I_{-j}} \geq \lambda^i \geq \max_{j=1,\ldots,l} \frac{R_j - R_0}{I_j} =: l.$$

Linear Progrmming Model: Dual

Using complementary slackness condition we can eliminate λ^c and write the dual problem as

$$d = \min_{\lambda^i \in [l,u]} \phi(\lambda^i),$$

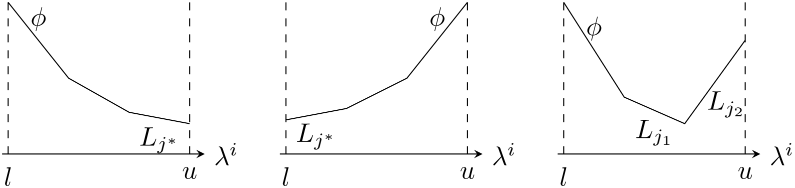
where $\phi(\lambda^i) = \max_{k=0,1,\ldots,n} L_k(\lambda^i)$, $L_0(\lambda^i) = i\lambda^i$ and

$$L_k(\lambda^i) := \frac{c(r_k - R_0)}{c_k} + \frac{ic_k - ci_k}{c_k}\lambda^i, k = 1, ..., n.$$

We note that each L_k represent an asset.

Linear Progrmming Model: Dual

Different behavior of ϕ yields the following possible configurations of the dula solutions.



(a) Decreasing.

(b) Increasing.

(c) Attains the minimum inside (l, u).

Figure: Illustration of ϕ.

Linear Model

We found an explicit form of solutions that can help guide the practical decision making:

- Only two assets hedging against each other in risks are needed in the optimal balance sheet.
- Dual solution indicates the value of the risks.
- Assets in the optimal balance sheet are determined by high risk adjusted returns.
- Thus, there is an efficient algorithm for constructing the optimal balance sheet.

Linear Model

- Involving more than two kind of risks is practically useful.
- The number of assets involved in the optimal balance sheet is the same as the number of risks.
- This is a nice extension of the pattern observed in the two-risk case and is related to the Caratheodory theorem.
- The characterization of the assets involved in the optimal balance sheet is much more involved and is a topic for investigation.

Ref. P. Júdice, M. Vazifedan and Q. J. Zhu, Bank Balance Sheet Allocation with interest, credit and liquidity risks, working paper.

Linear-Quadratic Model

- The concentration of the optimal portfolio in just a few assets in the linear model is impractical.
- This is due to the assumption that risks are linear – perfect correlation.
- The natural next step is to consider the correlation among risks for different assets –which leads to quadratic risk functions.
- The solution is easy if the quadratic risk function is positive definite.
- The realistic case involves semi-definite quadratic risk functions, that is challenging computationally for practical balance sheet management problems.

Ref. S. Maier-Paape, P. Júdice, A. Platen and Q. J. Zhu, Scalar and Vector Risk in the General Framework of Portfolio Theory, working book.

Duality

Duality plays an important role in bank balance sheet problems and in math finance in general

- Dual solution indicates marginal change of optimal value with respect to the corresponding risk limitation.
- Dual solution gives appropriate price of the insurance against risks.
- Dual constraints correspond to assets and indicates whether an asset is needed the optimal portfolio (binding).
- Duality helps solving the balance sheet problem.

Ref: P. Carr and Q. J. Zhu, Convex Duality and Financial Mathematics, Springer, 2018

From Gian-Carlo Rota

Richard Feynman was fond of giving the following advice on how to be a genius. You have to keep a dozen of your favorite problems constantly present in your mind, although by and large they will lay in a dormant state. Every time you hear or read a new trick or a new result, test it against each of your 12 problems to see whether it helps. Every once in a while there will be a hit, and people will say: "How did he do it? He must be a genius!"

Computation

Finding efficient algorithms for problems of real world scale is important for applications.

Hedging

Hedging with financial derivatives is an important topic that warrant further exploration.

Ref: F. Black and M. Scholes, The price of options and corporate liabilities, J. Polit. Econ. 81, 637-645, (1973)

Extreme Events

Analyzing risk mitigation using extreme value theory is an worthy direction.

Ref: A. J. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management, Princeton University Press, 2015

Conclusion

Applying various mathematical tools to the bank balance sheet management problem has been an interesting journey. This important yet concrete real world problem helps to

- add relevance to the research,
- provide new directions of investigation,
- serve as a check and validation tool for the theory, and
- stimulating new ideas.