The Method of Multiple Scales

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Multiscale Modeling

- Almost everything is multiscale
- Macroscale model: simpler & more efficient
- Microscale model: more accurate

Examples

- Chemical reactions
- Fluids (N-S good, but sometimes not)
- Turbulent flow
- Solids
- Mass distribution in the universe
- Rare events
- Faraday Cage (Trefethen)

Example problem

D. A. Edwards, "An Alternative Example of the Method of Multiple Scales", SIAM Review April 2000

1) The problem
2) Some things that don't work
3) Key idea -- Multiple time scale expansion

1) The Problem

Temperature in a rod that conducts heat,

Temperature = $u(x, t)$

where x = position along the rod,

t = time.

1) The Problem

Initially the rod has zero temperature everywhere,

$$u(x, 0) = 0$$

Starting at time $t = 0$, a specified heat flux is applied at one end of the rod:

$$u_x(0, t) = f(t)$$

1) The Problem

The heat flux applied at the end of the rod changes **very slowly** over time. To emphasize this write:

$$u_x(0, t) = f(\varepsilon t)$$

where ε is small and > 0.

1) The Problem

The heat equation is:

$$u_t = ku_{xx}$$

The other end of the rod is insulated:

$$u_x(L, t) = 0$$

1) The Problem

So the complete PDE problem is:

$$u_t = ku_{xx}$$

$$u(\mathrm{x}, 0) = 0$$

$$u_x(0, t) = f(\varepsilon t)$$

$$u_x(L, t) = 0$$

1) The Problem

Simplify by choosing units:

$$u_t = u_{xx}$$

$$u(\mathrm{x}, 0) = 0$$

$$u_x(0, t) = f(\varepsilon t)$$

$$u_x(1, t) = 0$$

Separation of Variables

Guess that $u(x, t)$ can be written in the form

$$u(x, t) = \varphi(x) \cdot g(t)$$

Then the heat equation $u_t = u_{xx}$

means

$$\frac{\partial}{\partial t}\Big(\varphi(x) \cdot g(t)\Big) = \frac{\partial^2}{\partial x^2}\Big(\varphi(x) \cdot g(t)\Big)$$

Separation of Variables

Guess that $u(x, t)$ can be written in the form

$$u(x, t) = \varphi(x) \cdot g(t)$$

Then the heat equation $u_t = u_{xx}$

means

$$\varphi(x) \cdot g'(t) = \varphi''(x) \cdot g(t)$$

Separation of Variables

Guess that $u(x, t)$ can be written in the form

$$u(x, t) = \varphi(x) \cdot g(t)$$

Then the heat equation $u_t = u_{xx}$

means

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{g'(t)}{g(t)}$$

Separation of Variables

Guess that $u(x, t)$ can be written in the form

$$u(x, t) = \varphi(x) \cdot g(t)$$

Then the heat equation $u_t = u_{xx}$

means

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{g'(t)}{g(t)} = -\lambda$$

Separation of Variables

$$\frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

$$\Rightarrow \quad \varphi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$\frac{g'(t)}{g(t)} = -\lambda$$

$$\Rightarrow \quad g(t) = Ce^{-\lambda t}$$

Separation of Variables

So *u* is of the form

$$u(x,t) = (Ce^{-\lambda t})\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right)$$

Separation of Variables

Using calculus (along with the IC & BC), get:

$$u(x,t) = \frac{g_0(t)}{2} + \sum_{n=1}^{\infty} g_n(t) \cdot \cos(n\pi x)$$

where

$$g_n(t) = -2e^{-n^2\pi^2 t} \cdot \int_0^t e^{n^2\pi^2 z} f(\varepsilon z)\, dz$$

First Idea -- Perturbation

Estimate heat flux:

$$f(\varepsilon t) = f(0) + \varepsilon t \cdot f'(0) + \cdots$$

And try to get

$$g_n(t;\varepsilon) = g_n^0(t) + \varepsilon g_n^1(t) + \cdots$$

First Idea -- Perturbation

Estimate heat flux:

$$f(\varepsilon t) = \boxed{f(0) + \varepsilon t \cdot f'(0)} + \frac{\varepsilon^2 t^2}{2} f''(0) + \cdots$$

And try to get

$$g_n(t;\varepsilon) = \boxed{g_n^0(t) + \varepsilon g_n^1(t)} + \frac{\varepsilon^2}{2} g_n^2(t) + \cdots$$

First Idea -- Perturbation

More calculations...

$$g_0^0(t) = -2t \cdot f(0)$$

$$g_0^1(t) = -t^2 \cdot f'(0)$$

So that the first term is

$$g_0(t) \sim -2t \cdot f(0) - \varepsilon t^2 \cdot f'(0)$$

Second Idea -- Rescale the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial x}(1,t) = 0$$

$$\frac{\partial u}{\partial x}(0,t) = f(\varepsilon t)$$

With t = "fast time"

Second Idea -- Rescale the problem

$$\varepsilon \frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial x}(1,T) = 0$$

$$\frac{\partial u}{\partial x}(0,T) = f(T)$$

With T = "slow time"

$T = \varepsilon t$

Separation of Variables

Again get:

$$u(x,T)=\frac{g_0(T)}{2}+\sum_{n=1}^{\infty}g_n(T)\cdot\cos(n\pi x)$$

Perturbation expansion

…

…

End up with

$$g_n^0(T) = \frac{-f(T)}{n^2\pi^2} \qquad \Rightarrow \qquad f(T) = 0$$

What didn’t work

- Fast time scale-based perturbation -> Model divergent over long time

- Slow time scale-based perturbation -> Inconsistency near $t = 0$

Multiple-Scale Expansion

- Key idea -- include both time scales in the model explicitly

$$u(x,t) = U(x,t,T)$$
$$T = \varepsilon t$$

Multiple-Scale Expansion

- Key idea -- include both time scales in the model explicitly

$$u(x,t) = U(x,t,T)$$
$$T = \varepsilon t$$

- Treat the two time variables as independent

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$$

Multiple time scale expansion

$$\frac{\partial U}{\partial t} + \varepsilon \frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial x^2}$$

$$U(x,0,0) = 0$$

$$\frac{\partial U}{\partial x}(1,t,T) = 0$$

$$\frac{\partial U}{\partial x}(0,t,T) = f(T)$$

Multiple time scale expansion

$$U(x,t,T) = \frac{G_0(t,T)}{2} + \sum_{n=1}^{\infty} G_n(t,T) \cdot \cos(n\pi x)$$

Normal issues

- Model macroscopic process
- Exact solution
- “Multiphysics”

Open research areas/problems

Weinan E, Bjorn Engquist, Xiantao Li, Weiqing Ren, Eric Vanden-Eijnden, "*The Heterogeneous Multiscale Method: A Review*" (2007)

- New application areas
- Error analysis
- Algorithm development