Mathematical Analysis of Investment Systems

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February 21, 2006

Abstract

Investment systems are studied using a framework that emphasize their *profiles* (the *cumulative probability distribution* on all the possible percentage gains of trades) and their *log return functions* (the expected average return per trade in logarithmic scale as a function of the investment size in terms of the percentage of the available capital). The *efficiency index* for an investment system, defined as the maximum of the log return function, is proposed as a measure to compare investment systems for their intrinsic merit. This efficiency index can be viewed as a generalization of Shannon's information rate for a communication channel. Applications are illustrated.

Key Words. trading systems, money management, entropy maximization, convex optimization, information rate.

1 Introduction

An investment system is usually a set of rules of buying and selling investment properties such as stocks, bonds, real estate, commodities and their derivatives for the purpose of capital appreciation. A common practice of evaluating and comparing the effectiveness of investment systems is to use their actual or simulated historical performances. We often see such comparisons for mutual funds in the surveys of financial periodicals and reports from financial institutions. Developers of trading systems will also provide simulation results for their methods (see e.g. [1, 3]). Academic research on certain investment methods can also be found (see e.g. [2, 5, 6]). However, these historical performances do not always reflect the true potential of investment systems because the results are often skewed by investment sizes. In the simulation of investment systems two popular methods are fixing a dollar amount for each trade and compounding the capital. It is well known that testing with a fixed dollar amount for each trade (equivalent to a simple algebraic sum of the percentage gain or loss of each trade) does not reflect the behavior of the investment system well. For example, suppose an investment system contains two trades losing and gaining the same percentage, say $r = p\%$.

trades	S1 %gain	S2 %gain	100% S1	100% S2	30% S1	30% S2
1	13%	6%	113.00	106.00	103.90	101.80
2	−25%	6%	84.75	112.36	96.11	103.63
3	13%	−5%	95.77	106.74	99.86	102.08
4	−25%	−5%	71.83	101.40	92.37	100.55
5	−25%	6%	53.87	107.49	85.44	102.36
6	13%	−5%	60.87	102.11	88.77	100.82
7	13%	−5%	68.79	97.00	92.23	99.31
8	13%	−5%	77.73	92.16	95.83	97.82
9	13%	6%	87.83	97.69	99.57	99.58
10	13%	6%	99.25	103.55	103.45	101.37

Table 1: Effects of investment systems under different investment sizes.

Then testing by a fixed dollar amount for each trade results in a gain of zero independent of the value of r. In contrast, compounding the investment capital results in a percentage gain $(1-r)(1+r)-1=-r^2$. That is to say, such an investment system always loses money and the greater the r the worse the loss. This relationship between the loss and the magnitude of r cannot be discovered by testing with a fixed dollar amount for each trade. Less obvious is that testing by compounding the capital is also subject to a similar pitfall.

Example 1.1. *We consider two simplified investment systems each with ten trades whose percentage gain (loss) is listed in the first two columns of Table 1. The effects of the two systems are tested using an investment capital of* \$100 *with two different investment sizes: 100% and 30% of the available capital for each trade, respectively. The results show that with an investment size of 100% of the available capital for each trade, System 2 is better than System 1, but with an investment size of 30% System 1 becomes better.*

Clearly, to compare the performance of investment systems, we need to focus on the percentage gain or loss for each trade and to take the impact of investment size into consideration. In this paper, we use a framework that characterizes investment systems with their *profile* – the *cumulative probability distribution* on all the possible percentage gains of trades. Based on the profile of an investment system we can establish its expected average return per trade in logarithmic scale as a function of the investment size in terms of the percentage of the available capital – the *log return function*. The log return function is a nice smooth concave function providing a comprehensive characterization of the behavior of the investment system. We define the *efficiency index* as the maximum of the log return function. As the average expected exponential growth rate per trade in logarithmic scale under the *best* investment size, the efficiency index is a good gauge for the comparison of investment systems.

One important application of the log return function and the efficiency index is to characterize invalid investment systems – those that will only lose money. As expected, the invalid investment systems are identified by a zero mathematical expectation of the profile. In practice an investment system has to at least outperform typical fixed income investments such as a certified deposit or a government bond to be useful. Exploration of investment systems that are invalid relative to a fixed income investment leads to a modified log return function that is concave but nonsmooth. Using this modified log return function, we show that an investment system is invalid with respect to a fixed income investment if and only if the absolute value of the mathematical expectation of the profile is bounded by the return of the corresponding fixed income investment. This result is used to analyze investment systems with parallel trades.

The study of the relationship between the investment size and the investment performance can be traced back to [4]. In this pioneering work Kelly gave an interpretation of Shannon's information rate for a communication channel with noise as the logarithm of the expected average exponential growth per bet when a gambler bets with the signals received from this communication channel under the best betting size. Kelly's work was developed by Thorp and Vince and was applied in betting and investing (see [13, 15, 16]). The results reported here are a further development along this line of research. The efficiency index can be viewed as a generalization of Shannon's information rate for a communication channel [12]. It measures the potential of an investment system in recognizing and utilizing (randomly occurring) trends of the price movements of the investment properties. Combined with the average annual turnover rate of the capital it provides an estimate of the annual percentage return of the investment system. In principle the idea used here can also be used to produce indicators for methods in capturing randomly occurring trends in other problems.

The rest of the paper is arranged as follows. In Section 2 we define an investment system and related concepts of profile, log return functions and efficiency index. We also discuss some of their properties. Section 3 discusses explicit computation formulae for efficiency indices of investment systems in two important cases. We also revisit Example 1.1, illustrating how to use the efficiency index and discussing its limitations. Section 4 discusses the relationship between the efficiency index for investment systems and Shannon's information rate for a communication channel with noise. Section 5 contains criteria for invalid investment systems. We then apply these criteria to discuss investment systems with parallel trades in Section 6. Section 7 contains the concluding remarks.

2 Investment systems, the log return function and the efficiency index

Let us consider the process of testing an investment system over a set of historical data. Denote the (finite) outcomes of the trades generated by the system in terms of percentage gain by $\{g_n : n = 1, \ldots, N\}$ with $g_1 < g_2 < \ldots < g_N$ ($g_n < 0$ represents a loss). Then the test will identify the frequency p_n associated with each outcome g_n. How good is this investment system? Besides the frequencies $\{p_n\}$, the return clearly also depends on the size of each trade as illustrated by the example in the introduction. Let us use s to denote the size of each trade as the percentage of the available capital and use M to denote the total number of trades in the test. Then the number of trades with gain g_n is Mp_n. Using $G(s)$ to denote the average exponential rate of growth of the investment capital per trade with a trading size s percent of the available capital we have $G(s)^M = \Pi_{n=1}^N (1 + sg_n)^{Mp_n}$ and

$$G(s) = \Pi_{n=1}^N (1 + sg_n)^{p_n}. \tag{2.1}$$

The maximum of $G(s)$ will give us a good indication of the potential profitability of the investment system. For the ease of analysis we will use its natural log

$$f(s) = \ln G(s) = \sum_{n=1}^{N} p_n \ln(1 + sg_n)$$

and call f the *log return function*. Since the natural log is an increasing function, the maximum of $f(s)$ will give us an equivalent indication of the effectiveness of the investment system. Normally, as a percentage of the available capital, the range of s should be $[0, 1]$. However, to explore the full potential of the investment system, we will allow s to take all the values in $(-1/g_N, -1/g_1)$, the domain of f, with the interpretation that $s > 1$ represents trade on margins and $s < 0$ represents shorts (both with idealized margin rate 0).

A closer look at the motivating example will convince us that the key to evaluate an investment system is the statistics on the outcomes of trades. How to generate those trades is unimportant for the comparison of investment systems. Thus, it suffices to view an investment system as a set of trades. Moreover, the concrete statistics over a set of historical data can be viewed as a sampling to determine the general probability distribution of the gains of the trades in the investment system. Now we formalize these observations.

Definition 2.1. (Investment system) *A* trade *is the process of acquiring an investment property with the investment capital and subsequently liquidating the property and returning the proceed to the investment capital. An* investment system $\mathcal{I}$ *is a set of (historical and/or future) trades. For each trade $T \in \mathcal{I}$ the outcome is measured by its percentage* gain *and is denoted by $g(T)$. The set $\mathcal{S}_{\mathcal{I}} = \{g(T) : T \in \mathcal{I}\}$ is called the* gain space *of the investment system $\mathcal{I}$. The* profile *P of an investment system $\mathcal{I}$ is the* cumulative probability distribution *on its gain space defined by*

$$P(x) = prob\{T \in \mathcal{I} : g(T) \le x\}.$$

Theoretically the gain of a trade in an investment system $\mathcal{I}$ can take any value in $(-\infty, \infty)$. In reality the possible percentage gains and losses of an investment system always belong to a finite interval. Thus, in what follows we always assume that $a = \inf \mathcal{S}_{\mathcal{I}}$ and $b = \sup \mathcal{S}_{\mathcal{I}}$ are finite. Clearly $P(x) = 0$ for $x \in (-\infty, a)$ and $P(x) = 1$ for $x \in (b, \infty)$. Let $a = g_0 < g_1 < \ldots < g_N = b$ be a partition of the interval $[a, b]$. Then, for a trade $T \in \mathcal{I}$, the probability of $g(T) \in [g_{n-1}, g_n)$ is $p_n = P(g_n) - P(g_{n-1})$. We can approximate the *expected* exponential rate of growth per trade of the investment system $\mathcal{I}$ under an investment size s by $G(s) = \Pi_{n=1}^{N}(1 + sg_n)^{p_n}$. In logarithmic scale we have

$$\ln G(s) = \sum_{n=1}^{N} \ln(1 + sg_n)[P(g_n) - P(g_{n-1})].$$

This is a Riemann-Stieltjes sum that converges to the log return function of the investment system when the partition becomes finer and finer.

Definition 2.2. (Log return function and efficiency index)*Let $\mathcal{I}$ be an investment system with a profile P and let $a = \inf \mathcal{S}_{\mathcal{I}}$ and $b = \sup \mathcal{S}_{\mathcal{I}}$. The* log return function *of $\mathcal{I}$ is defined by*

$$f_{\mathcal{I}}(s) = \int_a^b \ln(1 + sx)dP(x) \tag{2.2}$$

and the efficiency index *of $\mathcal{I}$ is defined by*

$$\gamma_{\mathcal{I}} = \sup_{s \in (-1/b, -1/a)} f(s). \tag{2.3}$$

Here the integration is in the sense of Riemann-Stieltjes. The subscript $\mathcal{I}$ will be omitted in the sequel unless doing so will cause confusion.

If $a \geq 0$ ($b \leq 0$) then when s approaches $+\infty$ ($-\infty$) the log return function $f(s)$ will approach $+\infty$. This corresponds to the unlikely case that the outcome of the trades are all gains (losses) and, of course, under our idealized environment of a zero margin interest rate one should long (short) on margins as much as one can and this leads to an unlimited average rate of return per trade. The interesting case is when $a < 0 < b$. Since $\ln(1 + sx)$ as a function of s is continuous and strictly concave for every x so is f on its domain $(-\frac{1}{b}, -\frac{1}{a})$. It is not hard to see that f approaches $-\infty$ when s approaches the endpoints of the interval $(-\frac{1}{b}, -\frac{1}{a})$. Thus, f actually attains its unique maximum at some $\bar{s} \in (-\frac{1}{b}, -\frac{1}{a})$. We can summarize the above discussion as

Proposition 2.1. (Characterization of finite efficiency indexes) *Let $\mathcal{I}$ be an investment system and let $a = \inf \mathcal{S}_{\mathcal{I}}$ and $b = \sup \mathcal{S}_{\mathcal{I}}$. Then $\gamma < \infty$ if and only if $0 \in (a, b)$. In this case there exists a unique best investment size $\bar{s} \in (-\frac{1}{b}, -\frac{1}{a})$ such that $\gamma = f(\bar{s})$.*

From now on we will always consider the interesting case when $\gamma < \infty$ unless explicitly stated otherwise.

Remark 2.1. The exponential of the log return function $G(s) := \exp(f(s))$ is the expected average rate of exponential growth per trade as a function of the investment size s. This function is also strictly concave since

$$\begin{aligned} G''(s) &= \exp(f(s))[(f'(s))^2 + f''(s)] \\ &= \exp(f(s)) \left[\left(\int_a^b \frac{x}{1+sx} dP(x) \right)^2 - \int_a^b \left(\frac{x}{1+sx} \right)^2 dP(x) \right] < 0 \end{aligned}$$

by the Jensen inequality (see [9, Section 5.5]).

Eliminating the influence of the investment size, the efficiency index provides a measure of the intrinsic merits of investment systems in recognizing and utilizing trends in the price movement of the investment properties. We can see that $G := \exp(\gamma)$ is the expected average rate of exponential growth per trade under the best investment size. Thus, the larger the γ the better the potential profitability of the investment system. Since $f(0) = 0$ we always have $\gamma \geq 0$ which implies that $G \geq 1$. In terms of investment this means that when a money making investment size cannot be found for an investment system one should not invest in it. A positive efficiency index indicates that the investment system can recognize trends of price movement and therefore has the *potential* of making money. However, this should not be confused with the original investment system *actually* makes money. A wrong investment size could turn a winning system to a losing one. Moreover, a positive efficiency index γ combined with a negative best investment size $\bar{s}$ indicates that the original investment system loses money. In other words, in terms of capturing trends it is consistently wrong. However, one could make money by using it in the opposite direction (i.e. short when the system recommends long and long when it recommends short). It turns out that the sign of the best investment size is the same as that of the mathematical expectation of the profile.

Proposition 2.2. (The sign of the best investment size)

$$\text{sgn } \bar{s} = \text{sgn } \mathcal{E}(P),$$

where sgn *is the* sign function *defined by*

$$\text{sgn } (s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if} s < 0. \end{cases}$$

Proof. Since f is a concave function we have (see [8])

$$0 \leq f(\bar{s}) = f(\bar{s}) - f(0) \leq f'(0)\bar{s}.$$

It follows that sgn $\bar{s} =$ sgn $(f'(0))$. Moreover we can directly calculate that $f'(0) = \int_a^b x dP(x)$. Since $P(x) = 0$ for $x \in (-\infty, a)$ and $P(x) = 1$ for $x \in (b, \infty)$ we have $f'(0) = \int_{-\infty}^{\infty} x dP(x) = \mathcal{E}(P)$, the mathematical expectation of the profile P. Q.E.D.

In practice the profile $P(x)$ has to be determined by sampling finite number of trades. This amounts to consider an investment system with a finite gain space $\{g_n : n = 1, \ldots, N\}$.

Proposition 2.3. (Investment system with a finite gain space) *The log return function and the efficiency index of an investment system with a finite gain space* $\{g_n : n = 1, \ldots, N\}$ *with* $(g_1 < g_2 < \ldots < g_N)$ *are*

$$f(s) = \sum_{n=1}^{N} p_n \ln(1 + s g_n), \tag{2.4}$$

and

$$\gamma = \max_{s \in (-\frac{1}{g_N}, -\frac{1}{g_1})} f(s). \tag{2.5}$$

Here p_n *is the probability for a trade to have a gain* g_n*.*

Proof. Observing that $P(x) = \sum_{k=1}^{n} p_k$ if $x \in [g_n, g_{n+1})$ for $n = 1, \ldots, N-1$ we have $f(s) = \sum_{n=1}^{N} p_n \ln(1 + s g_n)$. Q.E.D.

The best investment size was first discussed by Kelly in [4] for a gambling problem with a symmetric payoff. Thorp generalized Kelly's methods and applied them to gambling problems with asymmetric payoffs and to problems of portfolio choice [13, 14, 15]. The best investment size for investment systems with a finite gain space and a positive $\mathcal{E}(P)$ was discussed by Vince in [16]. Vince approached the problem by numerically searching for $\bar{s}$ using function $G(s)$ in several examples (with the domain of $G(s)$ scaled to the interval $[0, 1]$). He observed that a positive best investment size exists if and only if $\mathcal{E}(P) > 0$ and discussed how to approximate the best investment size $\bar{s}$.

The following theorem gives a procedure of calculating the efficiency index for an investment system with a finite gain space.

Theorem 2.1. (Compute efficiency index for investment systems with a finite gain space) *Consider an investment systems with a finite gain space $\{g_n : n = 1, \ldots, N\}$ with $g_1 < g_2 < \ldots < g_N$. Suppose that, for each n, p_n is the probability for a trade to have a gain g_n. Then the efficiency index for this investment system can be evaluated by*

$$\gamma = f(\bar{s}) = \sum_{n=1}^{N} p_n \ln(1 + \bar{s} g_n), \tag{2.6}$$

where $\bar{s}$ is the best investment size determined by the unique solution of the $(N-1)$th order polynomial equation

$$0 = \Pi_{n=1}^{N}(1 + s g_n) \left(\sum_{n=1}^{N} \frac{p_n g_n}{1 + s g_n} \right) \tag{2.7}$$

on the inverval $(-\frac{1}{g_N}, -\frac{1}{g_1})$.

Proof. Since the log return function,

$$f(s) = \sum_{n=1}^{N} p_n \ln(1 + s g_n),$$

is a strictly concave function on $(-\frac{1}{g_N}, -\frac{1}{g_1})$, its derivative is strictly decreasing. Moreover, it is easy to see that $\lim_{s \to (-1/g_N)+} f'(s) = \infty$ and $\lim_{s \to (-1/g_1)-} f'(s) = -\infty$. Thus, there is a unique solution $\bar{s}$ to the equation

$$0 = f'(s) = \sum_{n=1}^{N} \frac{p_n g_n}{1 + s g_n} \tag{2.8}$$

on $(-\frac{1}{g_N}, -\frac{1}{g_1})$ which is the best investment size and $\gamma = f(\bar{s})$.

Finally, observing that the polynomial $\Pi_{n=1}^{N}(1 + s g_n)$ has no solution in the interval $(-\frac{1}{g_N}, -\frac{1}{g_1})$ shows that $\bar{s}$ must be the unique solution of the $(N-1)$th polynomial equation

$$0 = \Pi_{n=1}^{N}(1 + s g_n) \left(\sum_{n=1}^{N} \frac{p_n g_n}{1 + s g_n} \right)$$

on the inverval $(-\frac{1}{g_N}, -\frac{1}{g_1})$. Q.E.D.

We can see that a closed form analytical formula for γ is not to be expect in general unless $N \leq 5$. Explicit formulae for the special cases when $N = 2$ and $N = 3$ are particularly useful and will be discussed in the next section. On the other hand a numerical estimate for γ can be derived either by numerically solving for $\bar{s}$ in (2.7) and then using $\gamma = f(\bar{s})$ or by directly searching for the optimal of the concave function f on the interval $(-\frac{1}{g_N}, -\frac{1}{g_1})$.

Finally, γ is related to the average rate of exponential gain per trade. Given an average number of trades per year, ν, the widely used annualized return for the investment system can be estimated as $\exp(\nu\gamma) - 1$. However, caution is warranted in using this number, since γ is derived under the idealized assumption that one can invest on margin and short without interest and restriction. When the investment system generates many simultaneous positions, it is hard to implement without leverage (see Section 6 for further analysis of investment systems with parallel trades and margin restrictions).

3 Investment systems with two and three distinct gains

The special cases of investment systems with two and three distinct gains are particularly important and warrant further analysis. They are related to the time tested strategy of cutting losses and taking profits at fixed thresholds that are often adopted by professional stock traders [7, 10]. Such an investment system works as follows: it has a set of rules for generating trading signals and two thresholds $c < 0$ and $t > 0$ in terms of percentage of cutting losses and taking profits, respectively. The investor will purchase a stock (using a predetermined position size) whenever a signal is generated. After the purchase, the investor will hold on to the stock until its price percentage change reaches either c or t. At which point the investor will liquidate that stock (assuming that is always possible). Then the investor will wait for the next signal and the above process will be repeated. Usually $|c|$ is much smaller than t and the idea is to cut losses quickly to preserve the capital and to let the winning trades cover the losses and make money. Clearly the outcome of a trade in such an investment system is either c or t and the system is characterized by the probability p_c and p_t of trades result in c and t, respectively.

Theorem 3.1. (Investment systems with two distinct gains) *Let $\mathcal{I}$ be an investment system with two distinct gains c and t with $c < t$. Suppose that the probabilities of a trade in $\mathcal{I}$ to have gains c and t are p_c and p_t, respectively. Then the best investment size and the efficiency index for $\mathcal{I}$ are*

$$\bar{s} = -\frac{p_c c + p_t t}{ct} \tag{3.1}$$

and

$$\gamma = p_c \ln \frac{p_c(t-c)}{t} + p_t \ln \frac{p_t(t-c)}{c} \tag{3.2}$$

Proof. The log return function for such an investment system is $f(s) = p_c \ln(1 + sc) + p_t \ln(1 + st)$. By Theorem 2.1, the best investment size $\bar{s}$ is the solution of equation

$$0 = (1 + sc)(1 + st)\left(\frac{p_c c}{1 + sc} + \frac{p_t t}{1 + st}\right).$$

Solving this equation produces equation (3.1). Then we can derive the efficiency index by $\gamma = f(\bar{s})$. Q.E.D.

One short-coming of the two threshold investment system described above is that sometimes the percentage change of the price of the stock fluctuates inside the interval (c, t) for a long time. This will tie up the investment capital and will result in inefficient utilization of the capital. Thus, a common practice is to set a maximum holding time. Using this method, a portion of the trades will exit when the maximum hold time is reached and result in gains or losses differing from t and c. One way to simplify the analysis of the modified system is to use the average gain for all such trades as the third possible outcome to estimate the performance of the system. Denoting the average gain of all the trades that exit without reaching c or t by a and denoting the corresponding probability by p_a, we have an investment system with three outcomes.

Theorem 3.2. (Investment systems with three distinct gains) *Let $\mathcal{I}$ be an investment system with three distinct gains c, a and t satisfying $c < a < t$. Suppose that the probabilities of a trade in $\mathcal{I}$ having gains c, a and t are p_c, p_a and p_t, respectively. Then the efficiency index for $\mathcal{I}$ is*

$$\gamma = p_c \ln(1 + c\bar{s}) + p_a \ln(1 + a\bar{s}) + p_t \ln(1 + t\bar{s}), \tag{3.3}$$

where $\bar{s}$ is the best investment size given by

$$\bar{s} = \begin{cases} 0 & \text{if } C = 0 \\ -\frac{p_c c + p_t t}{(p_c + p_t)ct} & \text{if } a = 0 \\ \frac{-B + \sqrt{B^2 - 4AC}}{2A} & \text{if } C < 0, a \neq 0 \\ \frac{-B - \sqrt{B^2 - 4AC}}{2A} & \text{if } C > 0, a \neq 0. \end{cases} \tag{3.4}$$

Here $A = tca$, $B = a[p_t t + p_c c + p_a(t + c)] + (p_t + p_c)tc$ and $C = p_t t + p_c c + p_a a$.

Proof. The proof is similar to that of Theorem 3.1. We omit the details. Q.E.D.

Let us re-examine Example 1.1 with the frameworks and tools developed in the past several sections. Both investment systems in Example 1.1 have two distinct gains and their profiles are summarized as follows.

	g_1	p_1	g_2	p_2
System 1	13%	0.7	−25%	0.3
System 2	6%	0.5	−5%	0.5

Drawing the log return functions of these two investment systems simultaneously in Figure 1 we can understand the reasons behind the phenomenon observed in Example 1.1. Moreover, we see that neither system was tested in Example 1.1 under the best investment size. Using Theorem 3.1 we can calculate that, for System 1, $\bar{s} = 49\%$, $\gamma = 0.040$ and for System 2, $\bar{s} = 167\%$, $\gamma = 0.041$. If we compare the efficiency indices only then the two investment systems are almost the same with System 2 slightly better. Yet this fact is hard to unveil without the help of the efficiency index. However, if margin is not allowed then System 1 is the better choice even though System 2 has a slightly higher efficiency index.

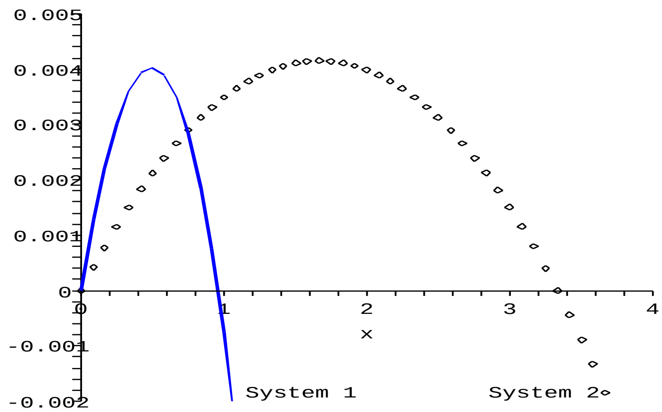

Figure 1: Log return functions

Examining the log return functions in Figure 1 we can concluded that one should never exceed the best investment size. The reason is one can always achieve the same return with a smaller investment size which, under the same investment system will lead to a smaller drawdown (an important feature of an investment system defined as the maximum percentage drop before recovery). Simulating the performance of these two investment systems under the best investment sizes we can find that for both systems, investing with the best investment sizes, the drawdown are near 30%. Such a drawdown is too big for most of the investors. Reducing the investment size will reduce both the return and the drawdown. However, the relationship is not linear: the drawdown declines faster than the return due to the concavity of $G(s) = \exp(f(s))$ (see Remark 2.1). Similar phenomena have been observed in simulations by experienced traders (see e.g. [11]). Thus, often it pays to invest with a size smaller than the best investment size.

4 Relationship with Shannon's information rate

Consider again the two state investment system related to fixed thresholds of cutting losses and taking profits. If the two thresholds for cutting losses and taking profits are symmetric in the sense that $c = -t$. Then the formula for calculating the efficiency index in (3.2) gives

$$\gamma = p_c \ln p_c + p_t \ln p_t + \ln 2.$$

Subject to a difference of a multiple of $\log_2 e$ this is Shannon's information rate for a communication channel with noise [12]. The difference is due to the choice of the bases in the log function and is not essential. Note that when $c = -1$ and $t = 1$ the investment system above is equivalent to a game with symmetric payoffs and we recovered what Kelly observed in [4]. In this sense we can regard the efficiency index in (3.2) as a generalization of Shannon's information rate of a communication channel with noise when the signal is used for a game with symmetric payoffs. In general the efficiency index indicates the potential profitability of an investment system with multiple (and possibly infinitely many) outcomes in a manner similar to Shannon's information rate which captures the potential of a communication channel with noise that has only two outcomes of transmitting a signal: correctly or incorrectly.

5 Invalid investment systems

If an investment system has an efficiency index $\gamma = 0$, then the unique best investment size is $\bar{s} = 0$ which corresponds to an exponential growth rate per trade $G = 1$. Then, for any investment size $s \neq 0$, $G(s) = exp(f(s)) < 1$. In other words anyone who invests in such a system will lose money. Such an investment system is invalid and should be avoided.

Definition 5.1. (Invalid investment systems) *An investment system is invalid if its efficiency index* $\gamma = 0$.

How can we characterize invalid investment systems? Let $\mathcal{I}$ be an invalid investment system with a profile P and a log return function

$$f(s) = \int_a^b \ln(1 + sx)dP(x),$$

where $a = \inf \mathcal{S}_{\mathcal{I}}$ and $b = \sup \mathcal{S}_{\mathcal{I}}$. Since $0 = f(0) = \gamma < \infty$ we must have $0 \in (a, b)$. Observing that $f(s)$ attains its maximum at $s = 0$, we have $f'(0) = \mathcal{E}(P) = 0$. The converse is also true because if $\mathcal{E}(P) = \int_a^b xdP(x) = 0$ holds then we must have $0 \in (a, b)$. Thus, $f'(0) = 0$ and $\gamma = f(0) = 0$. Summarizing we have the following.

Theorem 5.1. (Criteria for invalid investment systems) *An investment system with a profile* P *is invalid if and only if*

$$\mathcal{E}(P) = 0. \tag{5.1}$$

From Theorem 5.1 we can conclude that invalid investment systems are rare, since among all the profiles of possible investment systems the invalid investment systems can only be found in a particular hyperplane.

Definition 5.1 is a little idealized. In practice an investment system needs to at least outperform the usual fixed income investments such as a certified deposit or a government bond to be considered a valid option. Suppose that the benchmark interest rate for the available fixed income investment is i per trade. Then a return x becomes $x - i$ relative to this interest rate for $s > 0$ and $x + i$ for $s < 0$. Thus, relative to the fixed income investment with interest i per trade the log return function f needs to be replaced by

$$f_i(s) = \int_a^b \ln(1 + sx - |s|i)dP(x),$$

and an investment system is invalid relative to interest rate i if and only if

$$0 = f_i(0) = \max_{s \in (-\frac{1}{b+i}, -\frac{1}{a-i})} f_i(s). \tag{5.2}$$

Function f_i is concave but nonsmooth at $s = 0$. We can use the classical results in convex analysis (see [8]) to characterize (5.2). Here we give an elementary proof of the following theorem for completeness.

Theorem 5.2. (Criteria for relative invalid investment systems) *An investment system with a profile P is invalid relative to a fixed income interest rate i per trade if and only if*

$$|\mathcal{E}(P)| \leq i. \tag{5.3}$$

Proof. Using the Lagrange Mean Value Theorem we have

$$\ln(1+x) = \frac{x}{1+\theta(x)x}, \text{ where } \theta(x) \in (0,1). \tag{5.4}$$

Suppose that (5.2) holds. By (5.4), for $s > 0$, we have

$$0 \geq \frac{f_i(s) - f_i(0)}{s} = \int_a^b \frac{x-i}{1+\theta(s(x-i))s(x-i)} dP(x).$$

Taking limits as $s \to 0$, we have $0 \geq \int_a^b (x-i)dP(x) = \int_{-\infty}^{\infty}(x-i)dP(x)$ or $\int_{-\infty}^{\infty} xdP(x) \leq i$. Similarly, for $s < 0$, we have $\int_{-\infty}^{\infty} xdP(x) \geq -i$. It follows that

$$|\int_{-\infty}^{\infty} xdP(x)| \leq i.$$

Conversely, suppose (5.3) holds. Observing estimate (5.4) implies that $\ln(1+x) \leq x$ for all $x \in (-1, \infty)$, we have

$$\begin{aligned} f_i(s) &= \begin{cases} \int_a^b \ln(1+s(x-i))dP(x) & s \geq 0 \\ \int_a^b \ln(1+s(x+i))dP(x) & s < 0 \end{cases} \\ &\leq \begin{cases} \int_a^b s(x-i)dP(x) & s \geq 0 \\ \int_a^b s(x+i)dP(x) & s < 0 \end{cases} = \begin{cases} s(\int_{-\infty}^{\infty} xdP(x) - i) & s \geq 0 \\ s(\int_{-\infty}^{\infty} xdP(x) + i) & s < 0 \end{cases} \leq 0. \end{aligned}$$

Thus, (5.2) follows. Q.E.D.

We all know that the higher the performance requirement the less investment systems will be able to reach it. Theorem 5.2 is a quantitative form of this commonsense conclusion.

6 Investment systems with parallel trades

Now let us turn to investment systems that do not allow margins. Let $\mathcal{I}$ be such an investment system with a profile P. We assume $\mathcal{I}$ is profitable, that is $\mathcal{E}(P) > 0$. The contraposition of Theorem 5.2 tells us that $\mathcal{E}(P)$ is an upper bound for the return of the system per trade. Since $\exp(f(s))$ is a strict concave function, we have, for any $s \in [0,1]$,

$$\exp(f(s)) - \exp(f(0)) < \exp(f(0))f'(0)s = \mathcal{E}(P)s \leq \mathcal{E}(P).$$

That is to say, the return per trade $r(s) = \exp(f(s)) - 1$ under any investment size $s \in [0, 1]$ is strictly less than $\mathcal{E}(P)$. Of course this is assuming that we trade sequentially. If $\mathcal{I}$ has many parallel independent trades, can we do better by taking parallel trades? Let us consider the strategy of dividing the available capital into n equal parts and invest simultaneously in n different trades. Thus, for each *cycle* of the investment (a period for all the parallel trades to complete) we will have an expected investment return on the available capital given by $n(\exp(f(1/n)) - 1)$. The question is what is the best n to use. It turns out that the answer is the larger the n the better (neglecting the trading costs).

Theorem 6.1. *Let $\mathcal{I}$ be an investment system with a profile P. Assume that $\mathcal{E}(P) > 0$ and margin is not allowed. Further, we assume that the $\mathcal{I}$ has many parallel independent trades. Then investing all available capital into parallel trades equally, the return per trading cycle is an increasing function of the number of parallel trades approaching $\mathcal{E}(P)$ as the number of parallel trades increases.*

Proof. Let us denote $s = 1/n$. Then the expected return on each trade is $r(s) = \exp(f(s)) - 1$ and the total expected return per cycle is given by

$$R(s) = \frac{r(s)}{s}.$$

Observing that $r(s)$ is a strictly concave function by Remark 2.1 and $r(0) = 0$ we have

$$R'(s) = \frac{r'(s)s - r(s)}{s^2} = \frac{r'(s)s - r(s) + r(0)}{s^2} < 0,$$

that is to say, $R(s)$ is a decreasing function. Moreover,

$$\lim_{s \to 0+} R(s) = r'(0) = f'(0) = \mathcal{E}(P).$$

Q.E.D.

7 Concluding remarks

Investment systems are analyzed under a framework that emphasizes their statistical profile. The log return function is used to characterize investment systems. The efficiency index, the maximum of the log return function, is proposed as a measure for the potential profitability of investment systems. This concept can be viewed as a generalization of Shannon's information rate for a communication channel with noise. It gauges the potential of an investment system in recognizing and capturing trends in the price movement of the investment properties. Using this index allows us to avoid redundant simulation under different investment sizes in comparing investment systems. In particular, it leads to effective criteria for screening invalid investment systems (relative to a fixed income investment option) and an upper bound for the rate of return of the investment system given by the mathematical expectation of the profile. As an application, we discussed how to approach this upper bound in an investment system with many independent parallel trades. In general, the efficiency index is most useful in analyzing investment systems with trades that are mostly sequential while the mathematical expectation of the profile is an appropriate indicator of the potential profitability for investment systems with many independent parallel trades. The research reported here is focused on a general pattern and, therefore, we did not fully consider the impact of some factors such as investment costs, margin costs and drawdown. It is important to recognize that in designing actual investment systems, these are an important part of the analysis.

Acknowledgement. I thank the referee for his/her valuable comments.

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