Is Every Matrix Similar to a Toeplitz Matrix?

D. Steven Mackey* Niloufer Mackey† Srdjan Petrovic‡

21 February 1999

Submitted to Linear Algebra & its Applications

Abstract

We show that every $n \times n$ complex nonderogatory matrix is similar to a unique unit upper Hessenberg Toeplitz matrix. The proof is constructive, and can be adapted to nonderogatory matrices with entries in any field of characteristic zero or characteristic greater than n. We also prove that every $n \times n$ complex matrix with $n \leq 4$ is similar to a Toeplitz matrix.

Key words. Toeplitz, Hessenberg, canonical form, inverse problem, nonderogatory.

AMS subject classification. 15A21, 15A57

1 Introduction

The class of Toeplitz matrices is much studied, important within mathematics as well as in a wide range of applications. Such matrices arise, for example, in the theory of orthogonal polynomials, trigonometric moments, the design of stochastic filters, time series analysis, signal processing, and the analysis of the stability and convergence of solutions to initial-boundary-value problems for partial differential equations [7, 12, 15, 18, 19, 24].

Both the distribution of eigenvalues and the inverse eigenvalue problem for Toeplitz matrices have received much attention (see for example, [1, 3, 4, 6, 11, 20, 21, 22, 23]. The inverse eigenvalue question for real symmetric $n \times n$ Toeplitz matrices was posed in 1983 by Delsarte and Genin [6] and resolved by them for $n \leq 4$; the general case was settled only recently by Landau [17]. Landau's nonconstructive proof uses topological degree theory to show that any list of n real

*Department of Mathematics & Computer Science, Kalamazoo College, 1200 Academy Street, Kalamazoo MI 49006, USA. email: mackey@kzoo.edu.

†Department of Mathematics & Statistics, Western Michigan University, Kalamazoo, MI 49008, USA. email: nil.mackey@wmich.edu.

‡Department of Mathematics & Statistics, Western Michigan University, Kalamazoo, MI 49008, USA. email: srdjan.petrovic@wmich.edu.

numbers can be realized as the spectrum of an $n \times n$ real symmetric Toeplitz matrix.

By asking whether every complex matrix is similar to a Toeplitz one, we pose the inverse Jordan structure problem for Toeplitz matrices — which Jordan forms can be realized by some Toeplitz matrix? It is well-known that every diagonalizable matrix is similar to a Toeplitz matrix, indeed to a circulant matrix. This paper completely settles the question for the nonderogatory case in a constructive fashion. More precisely, given an $n \times n$ complex nonderogatory matrix A, we construct the unique upper Hessenberg Toeplitz matrix with ones on the subdiagonal that is similar to A. What happens for more general matrices? In what might seem to be a replay of the history of the inverse eigenvalue problem, we resolve the question when $n \leq 4$, showing that every 4×4 (or smaller) matrix is similar to a Toeplitz matrix. We also discuss some possible extensions and generalizations of these results.

2 Notation and Background Results

Toeplitz matrices are characterized by having constant diagonal entries. Formally, an $n \times n$ complex matrix A is Toeplitz if there exist $2n-1$ complex numbers $a_{-n+1}, \ldots, a_0, \ldots, a_{n-1}$ such that the ijth entry of A is a_{j-i} for $1 \leq i, j \leq n$.

The set of all $m \times n$ complex matrices will be denoted by $M_{mn}(\mathbb{C})$, or just $M_n(\mathbb{C})$, when $m = n$. As usual, $\lambda(A)$ is the spectrum of the matrix A, and we write $A \sim B$ to signify that matrices A and B are similar.

The following result is well-known [5, p. 66–73], and provides a context in which to view the results of this paper. We include a brief proof here for completeness.

Theorem 1 *Every diagonalizable matrix is similar to a Toeplitz matrix.*

Proof. It suffices to prove the result for diagonal matrices. Let Z be the cyclic permutation matrix

$$Z = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

Then Z is similar to $\operatorname{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1})$ where ω is a primitive nth root of unity. If $D = \operatorname{diag}(d_0, d_1, \ldots d_{n-1})$, then D is similar to a polynomial in Z. Specifically, let $p(z)$ be any polynomial such that $p(\omega^i) = d_i$ for $0 \leq i \leq n-1$, e.g., one can take $p(z)$ to be the Lagrange interpolating polynomial of degree $n-1$. Then D is similar to the Toeplitz (in fact, the circulant) matrix $p(Z)$. □

Recall that a matrix A is *upper Hessenberg* if all its entries below the first subdiagonal are zero, i.e., $a_{ij} = 0$ whenever $i > j + 1$. Extending the standard convention of referring to triangular matrices with ones on the main diagonal as unit triangular matrices, we will refer to upper Hessenberg matrices with ones on the first subdiagonal as *unit upper Hessenberg* matrices.

A matrix is said to be *nonderogatory* if all of its eigenspaces are one-dimensional. For the convenience of the reader, we briefly review in Proposition 1 some equivalent ways to characterize this class of matrices, beginning with a well-known result used to demonstrate one of the equivalences.

Lemma 1 *Let $A \in M_m(\mathbb{C})$, $B \in M_n(\mathbb{C})$ be such that $\lambda(A) \cap \lambda(B) = \emptyset$. Then for any $C \in M_{nm}(\mathbb{C})$,*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}.$$

Proof. Using a similarity by a matrix of the form $\left[\begin{smallmatrix} I & 0 \\ X & I \end{smallmatrix}\right]$, the result follows immediately from basic properties of Sylvester equations [16, p.270, 279]. $\square$

Proposition 1 *Let $A \in M_n(\mathbb{C})$. The following statements are equivalent:*

(a) *The matrix A is nonderogatory.*

(b) *In the Jordan form of A every eigenvalue of A appears in exactly one Jordan block.*

(c) *The minimal and the characteristic polynomials of A coincide.*

(d) *The set of matrices that commute with A is the same as the set of polynomials in A.*

(e) *The matrix A is similar to an upper Hessenberg matrix H with $h_{ij} \neq 0$ for $i = j + 1$.*

(f) *The matrix A is similar to a unit upper Hessenberg matrix.*

Proof. The equivalence of (a), (b) and (c) follows immediately from the Jordan canonical form. The equivalence of (d) with (a), (b), and (c) can also be deduced from the Jordan canonical form, but requires a little more argument. For details see [15, p. 135–7] or [16, p. 271–5]. To see that (b)⇒(f) first observe that every matrix in Jordan form is similar to its transpose, then use Lemma 1 repeatedly to fill in the missing 1's on the first subdiagonal. The implication (f)⇒(e) is trivial, so the proof will be complete once we establish that (e)⇒(a).

To that end, observe that $h_{ij} \neq 0$ for $i = j + 1$ together with $h_{ij} = 0$ for $i > j + 1$ implies that $\operatorname{rank}(H - \lambda I)$ is at least $n - 1$ for any $\lambda \in \mathbb{C}$. Hence $\operatorname{rank}(A - \lambda I) \geq n - 1$, and consequently the dimension of the kernel of $A - \lambda I$ cannot exceed 1. In other words, every eigenspace of A is one-dimensional, so A is nonderogatory. $\square$

We remark that upper Hessenberg matrices with the property that all entries on the first subdiagonal are non-zero are sometimes called *unreduced* Hessenberg matrices. The equivalence between nonderogatory matrices and unreduced upper Hessenberg matrices is discussed in [10, p. 367–9].

Since Toeplitz matrices are constant along diagonals, it will be natural and useful to consider matrices from a *diagonal perspective* [2], that is, to view them as sums of their diagonals. Following standard convention, the northwest-southeast diagonals of a matrix will be numbered $-(n-1), \ldots, 0, \ldots, n-1$, starting from the lower left corner. For each integer i, let $\Delta_i \subset M_n(\mathbb{C})$ denote the subspace of matrices whose non-zero entries are restricted to the ith diagonal:

$$\Delta_i = \{ A \in M_n(\mathbb{C}) : a_{rs} = 0 \quad \text{if} \quad s - r \neq i \} . \tag{1}$$

Note that if $|i| > n-1$ then Δ_i consists of just the zero matrix. It is now easy to see that

$$A \in \Delta_i \;\; \text{and} \;\; B \in \Delta_j \quad \Rightarrow \quad AB \in \Delta_{i+j} . \tag{2}$$

It will also be useful to have a compact notation for the matrix obtained from a given matrix $A \in \mathrm{M}_n(\mathbb{C})$ by filtering out all entries except those on the ith diagonal: $A^{(i)} \in \Delta_i$ is defined by

$$a_{rs}^{(i)} = \begin{cases} a_{rs} & \text{if } s - r = i \\ 0 & \text{otherwise} . \end{cases}$$

Now observe that any matrix A can be expressed as $A = \sum_i A^{(i)}$, the sum of its diagonals.

3 Existence of the Toeplitz canonical form

We prove our main theorem, that every nonderogatory matrix A is similar to a Toeplitz matrix, by establishing a stronger result — every nonderogatory matrix is similar to a unit upper Hessenberg Toeplitz matrix. Given the uniqueness result of the next section, it is reasonable to refer to this as the *Toeplitz canonical form* for a nonderogatory matrix.

The strategy for constructing this canonical form is simple and direct. Begin with *any* unit upper Hessenberg matrix H similar to A. Then using similarities by special unit upper triangular matrices, we can "fix" any selected diagonal of H, making all the entries along this diagonal the same, while keeping the lower diagonals unchanged. Working on one diagonal at a time in this manner, from lower left to upper right, we can eventually transform H into a Toeplitz matrix.

To carry out this strategy, we need some simple results about inverses of these special triangular matrices and similarities of upper Hessenberg matrices, viewed from the diagonal perspective referred to earlier.

Lemma 2 (a) *Let P be any unit upper triangular matrix with nonzero entries on at most two diagonals, i.e., $P = I + P^{(k)}$ for some $k \geq 1$. Then P^{-1} can have nonzero entries only on the ith diagonals where $i = 0,\ k,\ 2k,\ 3k,\ \ldots < n$.*

(b) *Suppose A is upper Hessenberg, and $B = P^{-1}AP$, where $P = I + P^{(k+1)}$ is unit upper triangular with $k \geq 0$. Then B is upper Hessenberg, and agrees with A up through the $(k-1)$st diagonal, i.e., $B^{(\ell)} = A^{(\ell)}$ for $\ell \leq k-1$. The kth diagonal of B (the first that could differ from A) is given by*

$$B^{(k)} = A^{(k)} + A^{(-1)}P^{(k+1)} - P^{(k+1)}A^{(-1)}\,.$$

Proof. (a) P^{-1} can be obtained by using the well-known result

$$(I+N)^{-1} = I - N + N^2 - \ \ldots\ + (-1)^{n-1}N^{n-1}$$

where N is any nilpotent. Since $\left(P^{(k)}\right)^j \in \Delta_{kj}$, it follows that P^{-1} can have nonzero entries only on diagonals with index jk for some $j \geq 0$.

(b) Using the result of part (a), we can expand B as

$$\begin{aligned} B &= P^{-1}A\,P = \left\{I - P^{(k+1)} + \sum_{m=2}^{n-1}(-1)^m \left(P^{(k+1)}\right)^m\right\} A \left\{I + P^{(k+1)}\right\} \\ &= A + A\,P^{(k+1)} - P^{(k+1)}A - P^{(k+1)}A\,P^{(k+1)} + \\ &\qquad \sum_{m=2}^{n-1}(-1)^m \left(P^{(k+1)}\right)^m A \left\{I + P^{(k+1)}\right\}. \end{aligned}$$

Since A is upper Hessenberg, we can write $A = A^{(-1)} + A^{(0)} + \ldots + A^{(n-1)}$. Recall that if $X \in \Delta_i$ and $Y \in \Delta_j$, then $XY \in \Delta_{i+j}$. Thus we see that nonzero contributions to $B^{(\ell)}$ for $\ell \leq k-1$ can come only from A, the first term in this expansion. Hence $B^{(\ell)} = A^{(\ell)}$ for $\ell \leq k-1$.

Observe that the fourth and fifth terms can contribute only to diagonals with index greater than or equal to $2k+1$. Hence the kth diagonal of B is built from just the first three terms:

$$\begin{aligned} B^{(k)} &= A^{(k)} + \left(A\,P^{(k+1)}\right)^{(k)} - \left(P^{(k+1)}A\right)^{(k)} \\ &= A^{(k)} + A^{(-1)}P^{(k+1)} - P^{(k+1)}A^{(-1)}\,. \qquad \square \end{aligned}$$

We are now in a position to prove the main result of this paper.

Theorem 2 *Every nonderogatory matrix in $M_n(\mathbb{C})$ is similar to a Toeplitz matrix, in particular to a unit upper Hessenberg Toeplitz matrix.*

Proof. Let $A \in M_n(\mathbb{C})$ be nonderogatory. By Proposition 1, we may assume without loss of generality that A is in unit upper Hessenberg form.

Now suppose P is a unit upper triangular matrix of the form $P = I + P^{(k+1)}$, where $0 \leq k \leq n-2$ is any fixed integer, and consider the matrix $B = P^{-1}A\,P$. We will show that for any unit upper Hessenberg A, it is always possible to choose $P^{(k+1)}$ so that the entries on the kth diagonal of B are *all equal*. Recall from Lemma 2(b) that $B^{(\ell)} = A^{(\ell)}$ for $\ell \leq k-1$, so the similarity $P^{-1}A\,P$ does not disturb any of the previous diagonals of A. Thus by applying this result $n-1$ times, starting at $k=0$ and working up to $k = n-2$ one diagonal at a time, we can transform any unit upper Hessenberg into a unit upper Hessenberg Toeplitz matrix.

All that remains is to see how to choose $P^{(k+1)}$ to achieve this goal. ¿From Lemma 2(b) we have

$$B^{(k)} = A^{(k)} + A^{(-1)}P^{(k+1)} - P^{(k+1)}A^{(-1)}\,, \tag{3}$$

where $A^{(-1)}$ is the fixed nilpotent matrix

$$A^{(-1)} = N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}. \tag{4}$$

Since $P^{(k+1)} \in \Delta_{k+1}$ can be chosen arbitrarily, we view it as an unknown X and analyze Eqn. (3) by studying the Sylvester operator

$$\begin{aligned} \mathcal{S} \;:\; \Delta_{k+1} &\longrightarrow \Delta_k \\ X &\mapsto NX - XN\,. \end{aligned}$$

Eqn. (3) then becomes $B^{(k)} = A^{(k)} + \mathcal{S}(P^{(k+1)})$, so characterizing the range of $\mathcal{S}$ will show which kth diagonals $B^{(k)}$ can be realized. Closely related questions have been addressed in [8], and we follow a similar approach here.

Consider the standard inner product defined on $M_n(\mathbb{C})$ by $\langle A, B\rangle = \mathrm{trace}(AB^H)$, where B^H denotes the conjugate transpose of B. By restriction this induces inner products on the subspaces Δ_{k+1} and Δ_k. With respect to these inner products, there is a well-defined adjoint map $\mathcal{S}^* : \Delta_k \longrightarrow \Delta_{k+1}$ such that $\langle \mathcal{S}(X), Y\rangle = \langle X, \mathcal{S}^*(Y)\rangle$ for all $X \in \Delta_{k+1}$ and $Y \in \Delta_k$. It is straightforward to check that $\mathcal{S}^*(Y) = N^H Y - Y N^H = N^T Y - Y N^T$. Now since $\mathrm{range}\,\mathcal{S} = (\ker \mathcal{S}^*)^\perp$, we may express Δ_k as the orthogonal direct sum

$$\Delta_k \;=\; \ker \mathcal{S}^* \oplus \mathrm{range}\,\mathcal{S}\,, \tag{5}$$

and compute $\mathrm{range}\,\mathcal{S}$ by first finding $\ker \mathcal{S}^*$. But $\mathcal{S}^*$ is just the restriction to Δ_k of the map

$$\begin{aligned} \widetilde{\mathcal{S}}^* \;:\; M_n(\mathbb{C}) &\longrightarrow M_n(\mathbb{C}) \\ Y &\mapsto N^T Y - Y N^T. \end{aligned}$$

Now the kernel of $\widetilde{\mathcal{S}}^*$ is well-known [9, 16], and can be found by direct computation, or, since N^T is nonderogatory, by invoking Proposition 1(d):

$$\ker \widetilde{\mathcal{S}}^* \;=\; \{n \times n \text{ upper triangular Toeplitz matrices}\}.$$

Thus we have

$$\ker \mathcal{S}^* \;=\; \Delta_k \cap \ker \widetilde{\mathcal{S}}^* \;=\; \{T \in \Delta_k : T \text{ is Toeplitz}\}\,,$$

and hence

$$\operatorname{range} \mathcal{S} = (\ker \mathcal{S}^*)^{\perp} = \{R \in \Delta_k : \text{ the sum of all the entries of } R \text{ is zero}\}.$$

Now we can see how to construct $P^{(k+1)}$. ¿From Eqn. (5) we know that $A^{(k)}$ can be uniquely decomposed as a sum $A^{(k)} = T + R$, where $T \in \ker \mathcal{S}^*$, and $R \in \operatorname{range} \mathcal{S}$. These components T and R are easily computed. Let α be the average of the entries on the kth diagonal of $A^{(k)}$. Then $T \in \Delta_k$ is the Toeplitz matrix with all α's on the kth diagonal, and $R = A^{(k)} - T$. Now let $P^{(k+1)}$ be the unique matrix in Δ_{k+1} such that $\mathcal{S}(P^{(k+1)}) = -R$. Again this is easily computed, since the matrix equation $\mathcal{S}(X) = -R$ reduces to a linear system of (scalar) equations that can be solved immediately by back-substitution. With this $P^{(k+1)}$ we have

$$B^{(k)} = A^{(k)} + \mathcal{S}(P^{(k+1)}) = T$$

is Toeplitz, and the proof is complete. $\square$

3.1 Extension to other fields

With some small modifications, the proof given above applies to nonderogatory matrices with entries in *almost* any field. Specifically, if F is any field of characteristic zero or characteristic greater than n, then any nonderogatory $A \in M_n(F)$ is similar to a unit upper Hessenberg Toeplitz matrix in $M_n(F)$.

Let us recall the main steps in the proof and briefly indicate where modifications are needed to accommodate F. The first step is to find some unit upper Hessenberg matrix similar to A; this was achieved in Proposition 1 for complex matrices via the Jordan canonical form. For matrices over an arbitrary field the Jordan form is unavailable. However, a characterization of nonderogatory matrices that plays a key role in the development of the rational canonical form [14, p. 227–237] provides a suitable replacement: A is nonderogatory if and only if A has a cyclic vector, equivalently, if and only if A is similar to the companion matrix of its characteristic polynomial. This companion matrix is then the desired unit upper Hessenberg matrix, and the proof now proceeds exactly as before, until we reach the matrix equation $B^{(k)} = A^{(k)} + \mathcal{S}(P^{(k+1)})$. The analysis of this equation can no longer rest on properties of inner products and adjoint operators, but fortunately a more concrete approach is still feasible. Viewing this matrix equation as a linear system of *scalar* equations shows that a Toeplitz diagonal $B^{(k)}$ can always be obtained, provided that division by the numbers $2, 3, \dots, n$ is always possible in F. This is where the condition on the characteristic of F originates, and simple counterexamples can be fashioned to show that it cannot be relaxed.

Indeed, for any field F of characteristic $p > 0$ and any integer $n \geq p$, there exists a nonderogatory matrix in $M_n(F)$ that is *not* similar to any unit upper Hessenberg Toeplitz matrix in $M_n(F)$. A specific example of such a matrix is given by the following. Let N be the nilpotent matrix displayed in Eqn. (4), and let E_{ij} denote the matrix with zeroes everywhere except for one in the ijth position. Also let kp be the largest multiple of p less than or equal to n. Then $B = N + E_{kp,n} \in M_n(F)$ is nonderogatory since it is the companion matrix of the polynomial $x^n - x^{kp-1}$. But, using Lemma 4 of the next section (which is valid in an arbitrary field), one can show that *no* unit upper Hessenberg Toeplitz matrix in $M_n(F)$ can have $x^n - x^{kp-1}$ as its characteristic polynomial. Thus B cannot be similar to any unit upper Hessenberg Toeplitz matrix in $M_n(F)$.

4 Uniqueness of the Toeplitz canonical form

We first establish a recurrence relation for the characteristic polynomials of unit upper Hessenberg Toeplitz matrices.

Lemma 3 *Let* $p_0(x) \equiv 1$, *and* $p_n(x) = \det(xI - T_n)$, *where*

$$T_n = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ 1 & a_1 & \ddots & \ddots & a_{n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ 0 & \dots & 0 & 1 & a_1 \end{pmatrix}, \quad n = 1, 2, \dots .$$

Then

$$p_n(x) = x p_{n-1}(x) - \sum_{i=1}^{n} a_i p_{n-i}(x), \quad n = 1, 2, \dots . \tag{6}$$

Proof. If we expand $p_n(x) = \det(xI - T_n)$ recursively by the first column, then at the mth stage we encounter the determinant $q_{m,n}(x)$ given by

$$q_{m,n}(x) = \begin{vmatrix} -a_m & -a_{m+1} & -a_{m+2} & \dots & \dots & -a_n \\ -1 & x - a_1 & -a_2 & \dots & \dots & -a_{n-m} \\ 0 & -1 & x - a_1 & \ddots & \ddots & -a_{n-m-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & x - a_1 & -a_2 \\ 0 & \dots & \dots & 0 & -1 & x - a_1 \end{vmatrix}.$$

Expanding $q_{m,n}(x)$ by the first column gives

$$q_{m,n}(x) = -a_m p_{n-m}(x) + q_{m+1,n}(x)$$

and iterating this formula yields

$$\begin{aligned} q_{m,n}(x) &= -a_m p_{n-m}(x) - a_{m+1} p_{n-m-1}(x) + q_{m+2,n}(x) \\ &\vdots \\ &= -a_m p_{n-m}(x) - a_{m+1} p_{n-m-1}(x) - \cdots - a_{n-2} p_2(x) + q_{n-1,n}(x). \end{aligned}$$

Since

$$q_{n-1,n}(x) = \begin{vmatrix} -a_{n-1} & -a_n \\ -1 & x - a_1 \end{vmatrix} = -a_{n-1}(x - a_1) - a_n = -a_{n-1} p_1(x) - a_n p_0(x)$$

we obtain that $q_{m,n}(x) = -\sum_{i=m}^{n} a_i p_{n-i}(x)$. In particular, $q_{2,n} = -\sum_{i=2}^{n} a_i p_{n-i}(x)$.

Now we expand $p_n(x)$ by its first column. This gives us

$$p_n(x) = (x - a_1) p_{n-1}(x) + q_{2,n}(x) = x p_{n-1}(x) - \sum_{i=1}^{n} a_i p_{n-i}(x)$$

and the lemma is proved. $\square$

The recurrence relation given in Eqn. (6) enables us to establish several useful facts about the coefficients of the characteristic polynomial $p_n(x)$ in terms of the entries of T_n. If we write

$$p_n(x) = x^n + c_{n1} x^{n-1} + c_{n2} x^{n-2} + \cdots + c_{n,n-1} x + c_{n,n} \tag{7}$$

then each coefficient c_{ni} is *a priori* a polynomial involving all the n complex variables $a_1, a_2, \ldots, a_n$. We now show that in fact, c_{ni} involves just the first i variables $a_1, a_2, \ldots, a_i$, and depends linearly on a_i.

Lemma 4 *If c_{ni} are the coefficients of the characteristic polynomial $p_n(x)$ as in (7), then*

$$c_{ni} = \begin{cases} -n a_1 & \text{if} \quad i = 1 \\ -(n - i + 1) a_i + d_{ni}(a_1, a_2, \ldots, a_{i-1}) & \text{if} \quad 2 \leq i \leq n \end{cases} \tag{8}$$

where $d_{ni}(a_1, a_2, \ldots, a_{i-1})$ is a polynomial in $a_1, a_2, \ldots, a_{i-1}$.

Proof. The proof is by induction on n, the degree of the polynomial $p_n(x)$.
Base Case: When $n = 1$,

$$p_1(x) = \det(xI - T_1) = x - a_1,$$

hence $c_{11} = -a_1$ and (8) holds. For $n = 2$, we have

$$p_2(x) = \det(xI - T_2) = \begin{vmatrix} x - a_1 & -a_2 \\ -1 & x - a_1 \end{vmatrix} = x^2 - 2a_1x + (-a_2 + a_1^2),$$

which gives $c_{21} = -2a_1$ and $c_{22} = -a_2 + a_1^2$. With $d_{22}(a_1) = a_1^2$, we see that (8) holds for $n = 2$.

Inductive Step: Assume c_{ki} satisfies (8) for each $k \leq n - 1$ and $1 \leq i \leq k$. We show that (8) remains valid when $k = n$ as well.

¿From the recurrence relation for $p_n(x)$ established in Lemma 3 we have

$$p_n(x) = xp_{n-1}(x) - \{a_1p_{n-1}(x) + a_2p_{n-2}(x) + \ldots + a_{n-1}p_1(x) + a_n\}. \qquad (9)$$

Equating the coefficients of x^{n-1} on both sides of Eqn. (9), and using the notation of Eqn. (7) we get

$$\begin{aligned} c_{n1} &= c_{n-1,1} - a_1 \\ &= -(n-1)a_1 - a_1 \quad \text{(by the induction hypothesis)} \\ &= -na_1 \end{aligned}$$

as required. Letting $2 \leq i \leq n$, we turn our attention to c_{ni}, the coefficient of x^{n-i} in $p_n(x)$. Which terms on the right hand side of (9) contribute to x^{n-i}? Since $p_k(x)$ is of degree k, such contributions can only come from

$$xp_{n-1}(x) - \{a_1p_{n-1}(x) + a_2p_{n-2}(x) + \ldots + a_{i-1}p_{n-i+1}(x) + a_ip_{n-i}(x)\}.$$

Equating the coefficients of x^{n-i} on both sides of (9) gives

$$\begin{aligned} c_{ni} &= c_{n-1,i} - \{a_1c_{n-1,i-1} + a_2c_{n-2,i-2} + \ldots + a_{i-1}c_{n-i+1,1} + a_i\} \\ &= \{c_{n-1,i} - a_i\} - \{a_1c_{n-1,i-1} + a_2c_{n-2,i-2} + \ldots + a_{i-1}c_{n-i+1,1}\} \\ &= \{X\} - \{Y\} \end{aligned}$$

Applying the inductive hypothesis to

$$c_{n-1,i-1}\,,\, c_{n-2,i-2}\,,\, \ldots\, c_{n-i+1,1}$$

we conclude that Y is a polynomial in the variables $a_1, a_2, \ldots, a_{i-1}$. Let us denote this polynomial by $d_{ni}(a_1, a_2, \ldots, a_{i-1})$.

Applying the inductive hypothesis to $c_{n-1,i}$ gives

$$X = c_{n-1,i} - a_i = -\left((n-1) - i + 1\right)a_i - a_i = -(n - i + 1)a_i\,.$$

Putting these facts together gives

$$c_{ni} = -(n - i + 1)a_i \ + \ d_{ni}(a_1, a_2, \ldots, a_{i-1})\,, \quad \text{for} \quad i = 2, \ldots, n$$

as desired. □

With Lemma 4 in hand, we are now able to prove uniqueness of the Toeplitz canonical form.

Theorem 3 *Every nonderogatory matrix in $M_n(\mathbb{C})$ is similar to a unique unit upper Hessenberg Toeplitz matrix .*

Proof. In light of Theorem 2 it suffices to show that if two unit upper Hessenberg Toeplitz matrices are similar, then they are equal. So let

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ 1 & a_1 & \ddots & \ddots & a_{n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ 0 & \dots & 0 & 1 & a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ 1 & b_1 & \ddots & \ddots & b_{n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_2 \\ 0 & \dots & 0 & 1 & b_1 \end{pmatrix},$$

and suppose that $A \sim B$. Then the characteristic polynomials of A and B must be the same. Equating their corresponding coefficients and using Lemma 4 we see that

$$-na_1 = -nb_1 \,, \tag{10}$$

and for $2 \leq k \leq n$,

$$-(n-k+1)a_k + d_{nk}(a_1, a_2, \dots, a_{k-1}) = -(n-k+1)b_k + d_{nk}(b_1, b_2, \dots, b_{k-1}) \,. \tag{11}$$

It is now easy to see that $a_k = b_k$ for $1 \leq k \leq n$ by induction on k. Clearly Eqn. (10) implies $a_1 = b_1$. Using the induction hypothesis, Eqn. (11) simplifies to $-(n-k+1)a_k = -(n-k+1)b_k$. Since $k \leq n$, we have $n-k+1 \neq 0$, and hence $a_k = b_k$. □

Remark 1 The proofs of Lemmas 3 and 4 are valid for nonderogatory matrices over an arbitrary field F. The proof of Theorem 3 remains valid for any field up to the very last line: there we need $n-k+1$ to be nonzero for $1 \leq k \leq n$, or equivalently, the numbers $1, 2, \dots, n$ must be nonzero in F. Thus the need to restrict the characteristic of F to either zero or greater than n is again made apparent.

Remark 2 The arguments in this section can be modified to give an alternative proof of the *existence* of the unit upper Hessenberg Toeplitz canonical form for nonderogatory matrices.

5 Similarity to Toeplitz: general case

For general matrices, we can answer the question posed in the title only for small n. We present our results here, showing that all 4×4 (or smaller) complex matrices are similar to Toeplitz matrices.

Theorem 4 *Every $A \in M_n(\mathbb{C})$ with $n \leq 4$ is similar to a Toeplitz matrix.*

Proof. We consider the cases $n = 2, 3, 4$ in turn.

$\boxed{n=2}$ The Jordan form of a 2×2 matrix can be one of the three types:

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq \beta.$$

Thus A is either diagonalizable or nonderogatory, and the result follows from Theorems 1 and 2.

$\boxed{n=3}$ Let A be a 3×3 matrix. When $|\lambda(A)| = 3$ or $|\lambda(A)| = 2$, the matrix A is either diagonalizable or nonderogatory, so the result follows from Theorem 1 or Theorem 2. It only remains to consider the case when $|\lambda(A)| = 1$. In this case the Jordan form of A can be written as $\alpha I + N$, where $\alpha \in \mathbb{C}$ and N is one of the following three nilpotent matrices:

$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{or} \qquad N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since N_0 and N_1 are Toeplitz, we turn our attention to N_3. A calculation shows that

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad SN_3S^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is clearly a Toeplitz matrix. This completes the proof when $n = 3$.

$\boxed{n=4}$ Not surprisingly, this is a bit more involved. Once again we distinguish several cases, depending on the cardinality of $\lambda(A)$, and focus our attention on those similarity classes not covered by either Theorem 1 or 2.

When $|\lambda(A)| = 4$ the matrix A is diagonalizable (as well as nonderogatory). If $|\lambda(A)| = 3$, say $\lambda(A) = \{\alpha, \beta, \gamma\}$, then there are just two different Jordan forms:

$$E_1 = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \beta & \\ & & & \gamma \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \beta & \\ & & & \gamma \end{pmatrix}.$$

So clearly, A is either diagonalizable or nonderogatory.

Consider next the case $|\lambda(A)| = 2$, say $\lambda(A) = \{\alpha, \beta\}$. Then A is similar to one of the following Jordan matrices:

$$E_3 = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \beta & \\ & & & \beta \end{pmatrix}, \qquad E_5 = \begin{pmatrix} \alpha & & & \\ & \beta & 1 & \\ & & \beta & 1 \\ & & & \beta \end{pmatrix}, \qquad E_7 = \begin{pmatrix} \alpha & & & \\ & \beta & 1 & \\ & & \beta & \\ & & & \beta \end{pmatrix},$$
$$E_4 = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \beta & \\ & & & \beta \end{pmatrix}, \qquad E_6 = \begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \beta & 1 \\ & & & \beta \end{pmatrix}, \qquad E_8 = \begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \beta & \\ & & & \beta \end{pmatrix}.$$

Clearly, E_3 and E_4 are diagonal matrices, E_5 and E_6 are nonderogatory, so it only remains to consider the situations when $A \sim E_7$ or $A \sim E_8$. Let

$$\widetilde{T}_7 = \begin{pmatrix} 4+2i & 4+4i & 3+6i & 1+8i \\ 4 & 4+2i & 4+4i & 3+6i \\ 4 & 4 & 4+2i & 4+4i \\ 0 & 4 & 4 & 4+2i \end{pmatrix}. \tag{12}$$

We will show that E_7 is similar to a Toeplitz matrix obtained by scaling and shifting $\widetilde{T}_7$. First, a similarity by the invertible matrix

$$S_7 = \begin{pmatrix} 16 & 24+8i & 24+16i & 20+28i \\ 0 & -6+16i & 10-16i & 2+i \\ 16 & -8-8i & -8 & -4-4i \\ -16 & 8+40i & 8-32i & 20+4i \end{pmatrix}$$

reveals the Jordan structure of $\widetilde{T}_7$:

$$S_7\,\widetilde{T}_7\,S_7^{-1} = \begin{pmatrix} 16+8i & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Next, a suitable scaling and shift produces a matrix that is clearly similar to E_7:

$$\frac{\alpha-\beta}{16+8i} S_7\,\widetilde{T}_7\,S_7^{-1} \;+\; \beta I = \begin{pmatrix} \alpha & & & \\ & \beta & \frac{\alpha-\beta}{16+8i} & \\ & & \beta & \\ & & & \beta \end{pmatrix}.$$

Finally, a diagonal similarity by $D = \operatorname{diag}(1, 16+8i, \alpha-\beta, 1)$, makes the $(2,3)$ entry equal to 1. Thus

$$E_7 \;=\; D\Big(\frac{\alpha-\beta}{16+8i} S_7\,\widetilde{T}_7\,S_7^{-1} \;+\; \beta I\Big)D^{-1} \;=\; D\,S_7\Big(\frac{\alpha-\beta}{16+8i}\,\widetilde{T}_7 \;+\; \beta I\Big)S_7^{-1}\,D^{-1},$$

so that E_7 is similar to the Toeplitz matrix $T_7 = \frac{\alpha-\beta}{16+8i}\,\widetilde{T}_7 + \beta I$.

The situation when $A \sim E_8$ can be handled in an analogous manner. Starting with the Toeplitz matrix $\widetilde{T}_8$ and the invertible matrix S_8 given by

$$\widetilde{T}_8 = \begin{pmatrix} 10 & 8 & 1 & -10 \\ 8 & 10 & 8 & 1 \\ 4 & 8 & 10 & 8 \\ 0 & 4 & 8 & 10 \end{pmatrix}, \qquad S_8 = \begin{pmatrix} 0 & 6 & 12 & 15 \\ 96 & 96 & 48 & -48 \\ 96 & -424 & 608 & -348 \\ -96 & 488 & -736 & 444 \end{pmatrix},$$

one can check that

$$S_8 \widetilde{T}_8 S_8^{-1} = \begin{pmatrix} 20 & 1 & & \\ & 20 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Hence a suitable scaling and shift of $\widetilde{T}_8$ yields a Toeplitz matrix T_8 that is similar to E_8.

We now turn our attention to the final case, $|\lambda(A)| = 1$. Let $\lambda(A) = \{\alpha\}$, so that $A = \alpha I + M$ where M is nilpotent. Clearly, it suffices to show that M is similar to a Toeplitz matrix. By inspection we see there are just five possibilities for the Jordan form of M:

$$M_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

that are already Toeplitz, and

$$M_3 = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Observe that M_3 and M_4 are both similar to Toeplitz matrices via permutation similarities:

$$M_3 \sim \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \qquad M_4 \sim \begin{pmatrix} 0 & & 1 & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Finally, a computation shows that if we let

$$T_5 = \begin{pmatrix} 0 & -2i & 2 & -1+2i \\ 4 & 0 & -2i & 2 \\ 8 & 4 & 0 & -2i \\ 16+8i & 8 & 4 & 0 \end{pmatrix}$$

and

$$S_5 = \begin{pmatrix} 0 & 2-2i & -1+i & 0 \\ 0 & -4+4i & -4-4i & 6-2i \\ 64 & 32-32i & 32 & -16+16i \\ -64 & -32-96i & -160 & 80-16i \end{pmatrix},$$

then $S_5\, T_5\, S_5^{-1} = M_5$, and the proof is complete. □

Remark 3 The matrices T_5, $\widetilde{T}_7$, and $\widetilde{T}_8$ were constructed by straightforward, but somewhat lengthy calculations. Starting with a generic 4×4 Toeplitz matrix T, it is possible in these particular cases to force the desired Jordan form on T simply by imposing conditions on the rank of T and on the coefficients of the characteristic polynomial of T. These conditions lead to equations constraining the entries of T, from which a parametrized family of Toeplitz matrices with the desired Jordan form can be generated. The particular matrices T_5, $\widetilde{T}_7$, and $\widetilde{T}_8$ are just convenient members of these families.

6 Beyond $n = 4$, and other extensions

Theorem 2 asserts that every nonderogatory matrix A is similar to some unit upper Hessenberg Toeplitz matrix. It is natural to wonder whether this result might be generalized to some class of "almost nonderogatory" matrices. In particular, one might consider classes of matrices defined by relaxing either condition (b) or (c) of Proposition 1. The former leads to the class of matrices whose eigenvalues have geometric multiplicity no larger than 2; equivalently, every eigenvalue appears in at most two Jordan blocks. Relaxing condition (c), on the other hand, gives rise to a slightly different class of matrices, those whose minimal and characteristic polynomials differ in degree by at most one. One might conjecture that if a matrix A belongs to one of these two classes, then A would be similar to some Toeplitz matrix with all zero entries below the *second* subdiagonal. Unfortunately, the following example shows that Theorem 2 cannot be extended in this way to either of these classes of "almost nonderogatory" matrices.

Example. Recall the nilpotent matrix

$$M = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

encountered in the previous section. Clearly, its characteristic polynomial is $p(\lambda) = \lambda^4$, its minimal polynomial is $m(\lambda) = \lambda^3$, and its only eigenvalue $\lambda = 0$ appears in 2 Jordan blocks. Therefore, M is "almost nonderogatory" in both senses, and belongs to each of the aforementioned classes.

In § 5 we displayed a Toeplitz matrix (T_5) similar to M. Notice that T_5 has nonzero entries on *every* diagonal except the main diagonal (any Toeplitz nilpotent must have zeroes on the main diagonal). Now if either of the two conjectured generalizations of Theorem 2 were true, then M would also be similar to a Toeplitz matrix of the form

$$T = \begin{pmatrix} 0 & b & c & d \\ a & 0 & b & c \\ 1 & a & 0 & b \\ 0 & 1 & a & 0 \end{pmatrix}.$$

(Because "scaling" similarities $D^{-1}TD$ with matrices of the form $D = \text{diag}(r, r^2, \ldots, r^n)$ preserve Toeplitz structure, we may assume without loss of generality that the lowest nonzero diagonal of T consists of all ones.) Since $M^3 = 0$, we must have $T^3 = 0$. A calculation shows that

$$T^3 = \begin{pmatrix} 2da + ca^2 + b^2 & a^2d + 3bc + 2b^2a & bd + 4bca + c^2 & 2dba + 2dc + c^2a + b^3 \\ 3ca + 2a^2b & 2b^2 + 2ca^2 + da & 2bc + 3b^2a + a^2d & bd + 4bca + c^2 \\ 4ba + c & 3a^2b + 2ca + d & 2b^2 + 2ca^2 + da & a^2d + 3bc + 2b^2a \\ a^3 + b & 4ba + c & 3ca + 2a^2b & 2da + ca^2 + b^2 \end{pmatrix}.$$

Successively setting the $(4, 1)$, $(3, 1)$, and $(3, 2)$ entries of T^3 to zero gives us the conditions $b = -a^3$, $c = 4a^4$, and $d = -5a^5$. Substituting into the $(3, 3)$ entry of T^3 we get

$$0 \;=\; 2b^2 + 2ca^2 + da \;=\; 2a^6 + 8a^6 - 5a^6 \;=\; 5a^6\,.$$

Thus $a = 0$ and hence, $b = c = d = 0$. So the matrix T must have the form

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

But now $T^2 = 0$, whereas $M^2 \neq 0$, so T cannot be similar to M, and the conjectures are both seen to be false.

Remark 4 The matrix M illustrates another important point. M is a real matrix, but it can be shown (as a corollary of the calculation outlined in Remark 3) that M is *not* similar to any *real* Toeplitz matrix. This is in contrast to the situation for real nonderogatory matrices, which are each guaranteed to be similar to some real Toeplitz matrix (see §3.1).

In spite of the failure of both conjectured generalizations of Theorem 2, it may still be possible to establish some useful Toeplitz structure theory for derogatory matrices. It is apparent, though, that Theorems 1 and 2 by themselves will not be sufficient to handle all large matrices. Already for $n = 4$ we are forced to deal with several instances when neither theorem applies: the Toeplitz matrices

T_5, T_7 and T_8 to which M_5, E_7, and E_8 are similar were found only after lengthy, albeit systematic calculations. Further computations in the same vein show that many 5×5 matrices, including all nilpotent ones, *are* similar to Toeplitz matrices. Although we have not found any 5×5 matrix that fails to be similar to a Toeplitz matrix, neither have we ruled out the possibility that such a counterexample exists.

Going far beyond $n = 4$, it is interesting to note that Theorem 4 does not extend to infinite dimensional Hilbert spaces. In this setting it is well-known that the only compact Toeplitz operator is the zero operator [13, p. 137]. But the set of compact operators is a two-sided ideal in the algebra of bounded linear operators on a Hilbert space [13, p. 85], so every operator similar to a compact operator must itself be compact. Consequently, no nonzero compact operator can be similar to a Toeplitz operator.

7 Conclusion

In this paper we have established a new Toeplitz canonical form for a large class of matrices — the nonderogatory matrices. Specifically, we have shown that every complex nonderogatory matrix is similar to a unique unit upper Hessenberg Toeplitz matrix. This is a first step towards solving the inverse Jordan structure problem for Toeplitz matrices — which Jordan canonical forms can be realized by some Toeplitz matrix? Put more simply, is every matrix similar to a Toeplitz matrix? The answer is yes for all 4×4 and smaller matrices, but the general question remains open.

References

[1] R. M. Beam and R. F. Warming, *The asymptotic spectra of banded Toeplitz and quasi-Toeplitz matrices*, SIAM J. Sci. Comput., 14 (1993), pp. 971–1006.

[2] E. C. Boman and M. A. Misconish, *A diagonal perspective on matrices*, College Math. J., 29 (1998), pp. 37–38.

[3] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*, Springer-Verlag, New York, 1998.

[4] G. Cybenko, *On the eigenstructure of Toeplitz matrices*, IEEE Trans. Acoustics, Speech and Signal Processing, ASSP-32 (1984), pp. 918–921.

[5] P. J. Davis, *Circulant Matrices*, John Wiley, New York, 1979.

[6] P. Delsarte and Y. Genin, *Spectral properties of finite Toeplitz matrices*, in Mathematical Theory of Networks and Systems: Proc. MTNS-83 Int.

Symp., Beer Sheva, Israel, June 1983, P. A. Fuhrmann, ed., vol. 58 of Lecture Notes in Control and Information Sciences, New York, 1984, Springer-Verlag, pp. 194–213.

[7] J. Durbin, *The fitting of time series models*, Revue de l'Institut International de Statistique, 28 (1960), pp. 233–43.

[8] H. Fassbender, D. S. Mackey, N. Mackey, and H. Xu, *Hamiltonian square roots of skew-Hamiltonian matrices*, Linear Algebra Appl., 287 (1999), pp. 125–159.

[9] F. R. Gantmacher, *The Theory of Matrices*, vol. I, Chelsea Publishing Co., New York, 1977.

[10] G. H. Golub and C. V. Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 2nd ed., 1989.

[11] R. M. Gray, *On the asymptotic eigenvalue distribution of Toeplitz matrices*, IEEE Trans. Info. Theory, 18 (1972), pp. 725–730.

[12] U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. of California Press, Berkeley, 1958.

[13] P. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1982.

[14] K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, Englewood Cliffs, N.J., 1971.

[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1990.

[16] ——, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[17] H. J. Landau, *The inverse eigenvalue problem for real symmetric Toeplitz matrices*, J. Amer. Math. Soc., 7 (1994), pp. 749–767.

[18] N. Levinson, *The Wiener RMS error criterion in filter design and prediction*, J. Math. Physics, 25 (1947), pp. 261–78.

[19] A. V. Oppenheim, *Applications of Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, 1978.

[20] L. Reichel and L. N. Trefethen, *Eigenvalues and pseudo-eigenvalues of Toeplitz matrices*, Linear Algebra Appl., 162–164 (1992), pp. 153–185.

[21] W. F. Trench, *On the eigenvalue problem for Toeplitz band matrices*, Linear Algebra Appl., 64 (1985), pp. 199–214.

[22] ——, *Interlacement of the even and odd spectra of real symmetric Toeplitz matrices*, Linear Algebra Appl., 195 (1993), pp. 59–69.

[23] ——, *Numerical solution of the inverse eigenvalue problem for real symmetric Toeplitz matrices*, SIAM J. Sci. Comp., 18 (1997), pp. 1722–1736.

[24] R. F. Warming and R. M. Beam, *An eigenvalue analysis of finite-difference approximations for hyperbolic IBVPs II: the auxiliary Dirichlet problem*, in Third International Conference on Hyperbolic Problems, B. Engquist and B. Gustafsson, eds., vol. II, Studentlitteratur, Lund, 1991, pp. 923–93.