Three Tutorial Lectures

Bruce E. Sagan
Michigan State University

Factoring Rook Polynomials

A *boardB* is a subset of the squares of an $n \times n$ chess board. Let $r_k(B)$ denote the number of ways of placing k non-attacking rooks on B which means that every row and column has at most one rook. Various specializations of $r_k(B)$ count permutations, derangements, and set partitions. One nice set of boards are the *Ferrers boards* $B = (b_1, \ldots, b_n)$ where the b_j are a weakly increasing sequence of nonnegative integers and the corresponding board is obtained by choosing the lowest b_j squares in column j of the chess board for $1 \leq j \leq n$. In a landmark paper, Goldman, Joichi, and White showed that if B is a Ferrers board then an appropriately chosen generating function for the $r_k(B)$ factors over the integers. They also gave various applications of this result, such as a new proof of a theorem of Foata and Schützenberger. This workshop will survey these beautiful results.

Introduction to Möbius Inversion over Posets

A *partially ordered set*, usually abbreviated to *poset*, is a set P together with an order relation where there can be elements of P which are not comparable by the relation. For example, we could take P to be all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. In this poset, $\{1, 2\}$ and $\{2, 3\}$ are incomparable because neither contains the other.

Möbius inversion is a powerful technique for inverting sums over posets. It has applications in combinatorics, number theory, and the finite difference calculus. In fact Gian-Carlo Rota, one of the fathers of modern combinatorics, considered this method so important that it is the focus of the very first paper in his series of articles *On the Foundations of Combinatorial Theory*.

This workshop will be an introduction to posets and their Möbius functions, including the fundamental Möbius Inversion Theorem. We will also discuss the characteristic polynomial of a poset which is the generating function for its Möbius function. In particular, we will see that the characteristic polynomial often factors over the integers.

Permutation Patterns and Statistics

Call two sequences of distinct integers $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ *order isomorphic* if they have the same pairwise comparisons, i.e., $a_i < a_j$ if and only if $b_i < b_j$ for all indices i, j. For example 132 and 475 are order isomorphic since both begin with the smallest element, have the largest element second, and end with the middle sized element. Let $\mathfrak{S}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$ viewed as sequences $\pi = a_1a_2\ldots a_n$. We say that $\sigma \in \mathfrak{S}_n$ contains $\pi \in \mathfrak{S}_k$ as a *pattern* if there is a subsequence σ' of σ which is order isomorphic to π. To illustrate, $\sigma = 6473521$ contains $\pi = 132$ as a pattern because of the subsequence $\sigma' = 475$. If σ does not contain π, we say it *avoids* π and use the notation

$$\mathrm{Av}_n(\pi) = \{\sigma \in \mathfrak{S}_n \ : \ \sigma \text{ avoids } \pi\}.$$

The theory of pattern containment and avoidance has recently become an object of intense study with many beautiful results as well as connections to computer science and algebraic geometry.

A *permutation statistic* is a function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$ where the range is the non-negative integers. Two famous statistics are the inversion number $\operatorname{inv}\sigma$, which counts the number of out-of-order pairs in σ. For example, $\sigma = 21453$ has inversions between the 2 and the 1, between the 4 the 3, and between the 5 and the 3, so that $\operatorname{inv}\sigma = 3$. Such statistics have a long and venerable history. In this workshop we will see how permutation patterns and statistics can be combined to obtain new and interesting results as well as shed light on old ones.